THE LARGEST CHARACTER DEGREES OF THE SYMMETRIC AND ALTERNATING GROUPS

ZOLTÁN HALASI, CAROLIN HANNUSCH, AND HUNG NGOC NGUYEN

Abstract. We show that the largest character degree of an alternating group A_n with $n \geq 5$ can be bounded in terms of smaller degrees in the sense that

$$b(\mathsf{A}_n)^2 < \sum_{\substack{\psi \in \mathrm{Irr}(\mathsf{A}_n) \\ \psi(1) < b(\mathsf{A}_n)}} \psi(1)^2,$$

where $\mathrm{Irr}(\mathsf{A}_n)$ and $b(\mathsf{A}_n)$ respectively denote the set of irreducible complex characters of A_n and the largest degree of a character in $\mathrm{Irr}(\mathsf{A}_n)$. This confirms a prediction of I. M. Isaacs for the alternating groups and answers a question of M. Larsen, G. Malle, and P. H. Tiep.

1. Introduction

For a finite group G, let $\mathrm{Irr}(G)$ and $b(G)$ respectively denote the set of irreducible complex characters of G and the largest degree of a character in $\mathrm{Irr}(G)$, then set

$$\varepsilon(G) := \frac{\sum_{\chi \in \mathrm{Irr}(G),\, \chi(1) < b(G)} \chi(1)^2}{b(G)^2}.$$

Since $b(G)$ divides $|G|$ and $b(G)^2 \leq |G|$, one can write $|G| = b(G)(b(G) + e)$ for some non-negative integer e. The (near-)extremal situations where $b(G)$ is very close to $\sqrt{|G|}$, or equivalently e is very small, have been studied considerably in the literature, see [Ber, Sny]. According to the result of Y. Berkovich [Ber] which says that $e = 1$ if and only if G is either an order 2 group or a 2-transitive Frobenius group, there is no upper bound for $|G|$ in this case. On the other hand, when $e > 1$, N. Snyder [Sny] showed that $|G|$ is bounded in terms of e and indeed $|G| \leq ((2e)!)^2$.

In an attempt to replace Snyder's factorial bound with a polynomial bound of the form Be^6 for some constant B, Isaacs [Isa] raised the question whether the largest character degree of a non-abelian simple group can be bounded in terms of smaller degrees in the sense that $\varepsilon(S) \geq \varepsilon$ for some universal constant $\varepsilon > 0$ and for all non-abelian simple groups S. Answering Isaacs's question in the affirmative, Larsen, Malle, and Tiep [LMT] showed

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Date: June 20, 2015.

2010 *Mathematics Subject Classification.* Primary 20C30, 20C15.

Key words and phrases. Symmetric groups, alternating groups, character degrees, largest character.

The research of the first author leading to these results has received funding from the European Union's Seventh Framework Programme (FP7/2007-2013) under grant agreement no. 318202, from ERC Limits of discrete structures Grant No. 617747 and from OTKA K84233. The third author is partially supported by NSA Young Investigator Grant #H98230-14-1-0293 and a BCAS Faculty Scholarship Award from the Buchtel College of Arts and Sciences, The University of Akron.

that the bounding constant ε can be taken to be $2/(120\,000!)$. We note that this rather small bound comes from the alternating groups, see [LMT, Theorem 2.1 and Corollary 2.2] for more details.

To further improve Snyder's bound from Be^6 to $e^6 + e^4$, Isaacs even predicted that $\varepsilon(S) > 1$ for every non-abelian simple group S. This was in fact confirmed in [LMT] for the majority of simple classical groups, and for all simple exceptional groups of Lie type as well as sporadic simple groups. Therefore, Larsen, Malle and Tiep questioned whether one can improve the bound $2/(120\,000!)$ for the remaining non-abelian simple groups – the alternating groups A_n of degree at least 5. Though Snyder's bound has been improved significantly by different methods in recent works of C. Durfee and S. Jensen [DJ] and M. L. Lewis [Lew], Isaacs's prediction and in particular Larsen-Malle-Tiep's question are still open.

In this paper we are able to show that $\varepsilon(\mathsf{A}_n) > 1$ for every $n \geq 5$.

Theorem 1.1. *For every integer $n \geq 5$,*

$$\sum_{\substack{\psi \in \mathrm{Irr}(\mathsf{A}_n) \\ \psi(1) < b(\mathsf{A}_n)}} \psi(1)^2 > b(\mathsf{A}_n)^2.$$

Unlike the simple groups of Lie type where one can use Lusztig's classification of their irreducible complex characters, it seems more difficult to work with the largest character degree of the alternating groups. For instance, while $b(S)$ is known for S a simple exceptional groups of Lie type or a simple classical group whose underlying field is sufficiently large (see [Sei, LMT]), $b(\mathsf{A}_n)$ as well as $b(\mathsf{S}_n)$ are far from determined. We note that the current best bound for $b(\mathsf{S}_n)$ is due to A. M. Vershik and S. V. Kerov [VK].

It is clear that $b(\mathsf{S}_n)/2 \leq b(\mathsf{A}_n) \leq b(\mathsf{S}_n)$ and as we will prove in Section 4, indeed $b(\mathsf{S}_n)/2 < b(\mathsf{A}_n) \leq b(\mathsf{S}_n)$ is always the case. As far as we know, it is still unknown for what n the equality $b(\mathsf{A}_n) = b(\mathsf{S}_n)$ actually occurs. It would be interesting to solve this. Though it appears at first sight that $b(\mathsf{A}_n) = b(\mathsf{S}_n)$ holds most of the time, computational evidence indicates that $b(\mathsf{A}_n) < b(\mathsf{S}_n)$ is true quite often.

When A_n and S_n do have the same largest character degree, Theorem 1.1 is indeed a direct consequence of a similar but stronger inequality for the symmetric groups.

Theorem 1.2. *For every integer $n \geq 7$,*

$$\sum_{\substack{\chi \in \mathrm{Irr}(\mathsf{S}_n) \\ \chi(1) < b(\mathsf{S}_n)}} \chi(1)^2 > 2b(\mathsf{S}_n)^2.$$

Our ideas to prove Theorems 1.1 and 1.2 are different from those in [LMT] and are described briefly as follows. We first introduce a graph with the partitions of n as vertices and a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ is connected by an edge to $\lambda_{dn} := (\lambda_1 + 1 \geq \lambda_2 \geq \ldots \geq \lambda_{k-1})$ only when $\lambda_k = 1$ and to $\lambda_{up} := (\lambda_1 - 1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 1)$ only when $\lambda_1 > \lambda_2$. It turns out that if λ corresponds to an irreducible character of S_n of the largest degree, then λ has precisely two neighbors in this graph. Furthermore, the degrees of the characters corresponding to λ_{up} and λ_{dn} are shown to be 'close' to that corresponding to

λ, see Lemma 3.4. With this in hand, we deduce that S_n has at least as many irreducible characters of degree close to but smaller than $b(\mathsf{S}_n)$ as those of degree $b(\mathsf{S}_n)$, and therefore Theorem 1.2 holds when the largest character degree $b(\mathsf{S}_n)$ has large enough multiplicity. When this multiplicity is smaller, we consider the irreducible constituents of the induced character $(\chi_{\mathsf{S}_{n-1}})^{\mathsf{S}_n}$ where $\chi \in \mathrm{Irr}(\mathsf{S}_n)$ is a character of degree $b(\mathsf{S}_n)$ and show that there are enough constituents of degree smaller than $b(\mathsf{S}_n)$ to prove the desired inequality.

As mentioned already, Theorem 1.1 follows from Theorem 1.2 in the case $b(\mathsf{A}_n) = b(\mathsf{S}_n)$. However, the other case $b(\mathsf{A}_n) < b(\mathsf{S}_n)$ creates some difficulties. To handle this, we first reduce the problem to the situation where S_n has precisely one irreducible character of degree $b(\mathsf{S}_n)$ and the second largest character degree equal to $b(\mathsf{A}_n)$. We then work with the multiplicity of degree $b(\mathsf{A}_n)$ and follow similar but more delicate arguments than in the case $b(\mathsf{A}_n) = b(\mathsf{S}_n)$.

Following the ideas outlined above, we can also prove the following

Theorem 1.3. *We have $\varepsilon(\mathsf{A}_n) \to \infty$ and $\varepsilon(\mathsf{S}_n) \to \infty$ as $n \to \infty$.*

This convinces us to believe that $\varepsilon(S) \to \infty$ as $|S| \to \infty$ for all non-abelian simple groups S and it would be interesting to confirm this.

The paper is organized as follows. In the next section, we give a brief summary of the representation/character theory of the symmetric and alternating groups. The graph on partitions and relevant results are presented in Section 3. Section 4 is devoted to the proofs of Theorems 1.1 and 1.2 and finally Theorem 1.3 is proved in Section 5.

To close this introduction, we mention that Theorem 1.1 has been used in the recent work [HLS] to obtain the best possible bound $|G| \leq e^4 - e^3$.

2. Preliminaries

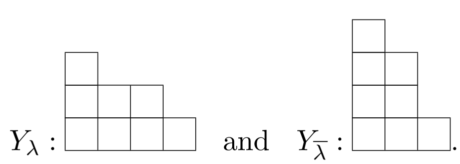
For the reader's convenience and to introduce notation, we briefly summarize some basic facts on the representation theory of the symmetric and alternating groups.

We say that a finite sequence of positive integers $\lambda := (\lambda_1, \lambda_2, \dots, \lambda_k)$ is a partition of n if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. The Young diagram corresponding to λ, denoted by Y_λ, is defined to be the finite subset of $\mathbb{N} \times \mathbb{N}$ such that

$$(i, j) \in Y_\lambda \text{ if and only if } i \leq \lambda_j.$$

The conjugate partition of λ, denoted by $\overline{\lambda}$, is the partition whose associated Young diagram is obtained from Y_λ by reflecting it about the line $y = x$. So $\lambda = \overline{\lambda}$ if and only if Y_λ is symmetric and in that case we say that λ is self-conjugate.

For example, if we take $n = 8$ with its partition $\lambda = (4, 3, 1)$, then $\overline{\lambda} = (3, 2, 2, 1)$ and their Young diagrams are



In our notation, the node $(i, j) \in Y_\lambda$ represents the box of the Young diagram Y_λ with horizontal coordinate i and vertical coordinate j.

For each node $(i,j) \in Y_\lambda$, the so-called *hook length* $h_\lambda(i,j)$ is defined by

$$h_\lambda(i,j) := 1 + \lambda_j + \overline{\lambda}_i - i - j.$$

That is, $h_\lambda(i,j)$ is the number of nodes that are directly above it, directly to the right of it, or equal to it. The *hook-length product* of λ is then defined by

$$H(\lambda) := \prod_{(i,j)\in Y_\lambda} h_\lambda(i,j).$$

For each positive integer n, it is known that there is a one-to-one correspondence between the irreducible complex characters of the symmetric group S_n and the partitions of n. We denote by χ_λ the irreducible character of S_n corresponding to λ. The degree of χ_λ is given by the *hook-length formula*, see [FRT]:

$$\chi_\lambda(1) = \frac{n!}{H(\lambda)}.$$

The irreducible characters of A_n can be obtained by restricting those of S_n to A_n. More explicitly, if λ is not self-conjugate then $(\chi_\lambda)_{\mathsf{A}_n} = (\chi_{\overline{\lambda}})_{\mathsf{A}_n}$ is irreducible and otherwise, $(\chi_\lambda)_{\mathsf{A}_n}$ splits into two different irreducible characters of the same degree. Therefore, the degrees of the irreducible characters of A_n labeled by λ are

$$\begin{cases} \chi_\lambda(1) & \text{if } \lambda \neq \overline{\lambda}, \\ \chi_\lambda(1)/2 & \text{if } \lambda = \overline{\lambda}. \end{cases}$$

For each partition λ of n, let $A(\lambda)$ and $R(\lambda)$ denote the sets of nodes that can be respectively added or removed from Y_λ to obtain another Young diagram corresponding to a certain partition of $n+1$ or $n-1$ respectively. As shown in [LMT, page 67] (see also [CHMN, §5]), we have $|A(\lambda)|^2 - |A(\lambda)| \leq 2n$, and hence

$$A(\lambda) \leq \frac{1+\sqrt{1+8n}}{2}.$$

Similarly, we have $|R(\lambda)|^2 + |R(\lambda)| \leq 2n$ and

$$R(\lambda) \leq \frac{-1+\sqrt{1+8n}}{2}.$$

The well-known branching rule (see [Jam, §9.2] for instance) asserts that the restriction of χ_λ to S_{n-1} is a sum of irreducible characters of the form χ_μ, where $Y_\mu = Y_\lambda \backslash \{(i,j)\}$ as (i,j) goes over all nodes in $R(\lambda)$. Also, by Frobenius reciprocity, the induction of χ_λ to S_{n+1} is a sum of irreducible characters of the form Y_ν, where $Y_\nu = Y_\lambda \cup \{(i,j)\}$ as (i,j) goes over all nodes in $A(\lambda)$.

It follows from the branching rule that the number of irreducible constituents of the induced characters $((\chi_\lambda)_{\mathsf{S}_{n-1}})^{\mathsf{S}_n}$ is at most

$$\frac{-1+\sqrt{1+8n}}{2} \cdot \frac{1+\sqrt{1+8(n-1)}}{2}.$$

In particular, this number is smaller than $2n$.

3. A graph on partitions

Let $\mathscr{P}$ denote the set of partitions of n. Furthermore, let

$$b_1 = b(\mathsf{S}_n) > b_2 > \ldots > b_m = 1$$

be the distinct character degrees of S_n. For every $1 \leq i \leq m$ let

$$\mathscr{M}_i := \{\lambda \in \mathscr{P} \mid \chi_\lambda(1) = b_i\}$$

so that

$$\mathscr{P} = \mathscr{M}_1 \cup \mathscr{M}_2 \cup \ldots \cup \mathscr{M}_m \quad \text{(disjoint union)}.$$

Definition 3.1. *For a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k)$ we define partitions λ_{up} and λ_{dn} in the following way. The partition λ_{up} is defined only if $\lambda_1 > \lambda_2$ and in this case let $\lambda_{up} := (\lambda_1 - 1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 1)$. Similarly, the partition λ_{dn} is defined only if $\lambda_k = 1$ and in this case let $\lambda_{dn} := (\lambda_1 + 1 \geq \lambda_2 \geq \ldots \geq \lambda_{k-1})$.*

Next we define a graph on $\mathscr{P}$.

Definition 3.2. *Let $\Gamma = (V, E)$ be the graph with vertex set $V = \mathscr{P}$ and edge set $E = \{(\lambda, \mu) \mid \mu = \lambda_{dn} \text{ or } \mu = \lambda_{up}\}$. Furthermore, let $\Gamma_{\mathscr{M}_i}$ be the induced subgraph of Γ on $\mathscr{M}_i$.*

For each vertex $\lambda \in V$, let $d(\lambda)$ denote the degree of λ, that is, the number of vertices that are connected to λ by an edge of Γ. It is clear that $d(\lambda) \leq 2$ for every $\lambda \in V$. Moreover, every connected component of Γ is a simple path.

Lemma 3.3. *For every $1 \leq r \leq m$ we have*

$$|\{\lambda \in \mathscr{M}_r \mid d(\lambda) < 2\}| \leq 2|\cup_{i<r} \mathscr{M}_i|.$$

In particular, we have the following

(1) $d(\lambda) = 2$ for all partitions $\lambda \in \mathscr{M}_1$.
(2) If $|\mathscr{M}_1| = 1$, then $d(\lambda) = 2$ for all but at most two partitions $\lambda \in \mathscr{M}_2$.

Proof. First we prove that

$$|\{\lambda \in \mathscr{M}_r \mid \nexists\, \lambda_{up}\}| \leq |\cup_{i<r} \mathscr{M}_i|.$$

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k) \in \mathscr{M}_r$ such that $\lambda_1 = \lambda_2 = \ldots = \lambda_s = t$ but $\lambda_{s+1} < t$ for some $1 < s \leq k$. Let $\lambda_{\to 1} := (\lambda_1 + 1, \lambda_2, ..., \lambda_s - 1, \lambda_{s+1}, ..., \lambda_k)$ and $x_i := h_\lambda(i, s)$ for every $1 \leq i \leq t-1$. Calculating the ratio of the hook-length products $H(\lambda_{\to 1})$ and $H(\lambda)$ and noting that $h_{\lambda_{\to 1}}(t, 1) = h_\lambda(t, 1) = s$, we get

$$\begin{aligned}
\frac{H(\lambda_{\to 1})}{H(\lambda)} &= \frac{\prod_{i=1}^{t-1}(h_{\lambda_{\to 1}}(i,s) h_{\lambda_{\to 1}}(i,1)) \prod_{j=1}^{s-1} h_{\lambda_{\to 1}}(t,j)}{\prod_{i=1}^{t-1}(h_\lambda(i,s) h_\lambda(i,1)) \prod_{j=1}^{s-1} h_\lambda(t,j)} \\
&= \frac{\prod_{i=1}^{t-1}((x_i - 1)(x_i + s)) \cdot (s-2)!}{\prod_{i=1}^{t-1}(x_i(x_i + s - 1)) \cdot (s-1)!} \\
&= \frac{1}{s-1} \prod_{i=1}^{t} \left(1 - \frac{s}{x_i(x_i + s - 1)}\right) < 1.
\end{aligned}$$

Hence for the degrees of characters we get

$$\frac{\chi_\lambda(1)}{\chi_{\lambda_{\to 1}(1)}} = \frac{H(\lambda_{\to 1})}{H(\lambda)} < 1.$$

Thus, we have defined a map $\lambda \mapsto \lambda_{\to 1}$ from the set $\{\lambda \in \mathscr{M}_r \mid \nexists\ \lambda_{up}\}$ into $\cup_{i<r}\mathscr{M}_i$. This map is clearly injective, so

$$|\{\lambda \in \mathscr{M}_r \mid \nexists\ \lambda_{up}\}| \le |\cup_{i<r} \mathscr{M}_i|$$

follows. The dual map $\lambda \mapsto \overline{\lambda}$ defines a bijection between $\{\lambda \in \mathscr{M}_r \mid \nexists\ \lambda_{up}\}$ and $\{\lambda \in \mathscr{M}_r \mid \nexists\ \lambda_{dn}\}$. It follows that

$$|\{\lambda \in \mathscr{M}_r \mid \nexists\ \lambda_{dn}\}| \le |\cup_{i<r} \mathscr{M}_i|.$$

Therefore,

$$|\{\lambda \in \mathscr{M}_r \mid d(\lambda) < 2\}| \le |\{\lambda \in \mathscr{M}_r \mid \nexists\ \lambda_{dn}\}| + |\{\lambda \in \mathscr{M}_r \mid \nexists\ \lambda_{up}\}| \le 2|\cup_{i<r} \mathscr{M}_i|$$

and the proof is complete. □

Lemma 3.4. *If* $d(\lambda) = 2$ *then*

$$1 < \frac{H(\lambda_{dn})H(\lambda_{up})}{H(\lambda)^2} < 4.$$

Proof. Let $\lambda = (\lambda_1 > \lambda_2 \ge \ldots \ge \lambda_k = 1) \in \mathscr{P}$ with $d(\lambda) = 2$. Furthermore, let $x_i := h(i, 1)$ for $2 \le i \le \lambda_1 - 1$ and $y_j := h(1, j)$ for $2 \le j \le k - 1$. Then we have $x_2 > x_3 > \ldots > x_{\lambda_1-1} \ge 2$ and $y_2 > y_3 > \ldots > y_{k-1} \ge 2$.

Calculating the ratios $H(\lambda_{up})/H(\lambda)$ and $H(\lambda_{dn})/H(\lambda)$ we obtain

$$\frac{H(\lambda_{up})}{H(\lambda)} = \prod_{i=2}^{\lambda_1-1} \frac{x_i - 1}{x_i} \cdot 2 \prod_{j=2}^{k-1} \frac{y_j + 1}{y_j}$$

and

$$\frac{H(\lambda_{dn})}{H(\lambda)} = 2 \prod_{i=2}^{\lambda_1-1} \frac{x_i + 1}{x_i} \cdot \prod_{j=2}^{k-1} \frac{y_j - 1}{y_j}.$$

It follows that

$$\frac{H(\lambda_{dn})H(\lambda_{up})}{H(\lambda)^2} = 4 \prod_{i=2}^{\lambda_1-1} \frac{x_i^2 - 1}{x_i^2} \prod_{j=2}^{k-1} \frac{y_j^2 - 1}{y_j^2}.$$

The right hand side of this inequality is clearly smaller than 4. Regarding the lower bound, we argue as follows. First, since the hook lengths x_i are different integers bigger than 1, we have

$$2 \prod_{i=2}^{\lambda_1-1} \frac{x_i^2 - 1}{x_i^2} > 2 \prod_{m=2}^{\infty} \frac{(m-1)(m+1)}{m^2} = 1.$$

The same can be said about $2 \prod_{j=2}^{k-1} \frac{y_j^2-1}{y_j^2}$ and so their product is also bigger than 1. The proof is complete. □

Using the previous lemma, we can show that S_n has many irreducible characters of degree close to but smaller than $b(\mathsf{S}_n)$.

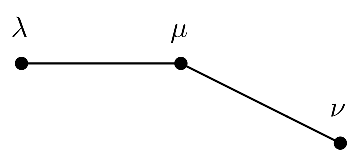
Proposition 3.5. *For every $1 \le r \le m$ we have*
$$\left|\left\{\mu \in \mathscr{P} \,\middle|\, \frac{b_r}{4} < \chi_\mu(1) < b_r\right\}\right| \ge |\mathscr{M}_r| - 4|\cup_{i<r} \mathscr{M}_i|.$$
In particular, we have

(1)
$$\left|\left\{\mu \in \mathscr{P} \,\middle|\, \frac{b_1}{4} < \chi_\mu(1) < b_1\right\}\right| \ge |\mathscr{M}_1|.$$
(2) If $|\mathscr{M}_1| = 1$, then
$$\left|\left\{\mu \in \mathscr{P} \,\middle|\, \frac{b_2}{4} < \chi_\mu(1) < b_2\right\}\right| \ge |\mathscr{M}_2| - 4.$$

Proof. For a real-valued function $f : V \mapsto \mathbb{R}$ defined on the vertex set of the graph $\Gamma = (V, E)$ we say that $x \in V$ is a local maximum (resp. minimum) of f if $f(y) \le f(x)$ (resp. $f(y) \ge f(x)$) for every $(x, y) \in E$.

Let C be any connected component of Γ, so C is a simple path. We note that if $d(\lambda) = 2$ then either $H(\lambda) < H(\lambda_{dn})$ or $H(\lambda) < H(\lambda_{up})$ by Lemma 3.4. Therefore, there is no local maximum $\lambda \in C$ of the hook-length product function $H : C \mapsto \mathbb{N}$ with $d(\lambda) = 2$. It follows that if $\lambda, \mu \in C$ are both local minima of H on C, then H is constant on the subpath connecting λ and μ. Furthermore, the restriction of H to a subpath of C of length ≥ 3 cannot be constant, since for an inner point λ of such a subpath we would have
$$\frac{H(\lambda_{dn})H(\lambda_{up})}{H(\lambda)^2} = 1$$
and this violates the inequality in Lemma 3.4. It follows from this argument that $|C \cap \mathscr{M}_r| \le 2$. Furthermore, if $|C \cap \mathscr{M}_r| = 2$, then the two vertices of $C \cap \mathscr{M}_r$ are either neighboring vertices in Γ or all the inner points of the subpath connecting them are elements from the set $\cup_{i<r}\mathscr{M}_i$.

We now introduce the notion of "level" in the graph Γ to represent the order of values taken by H on various partitions in a given connected component of the graph.[1] If $H(\lambda) > H(\mu)$ then the level of λ is higher than that of μ. For example, if $C = \{\lambda, \mu, \nu\}$, then the figure

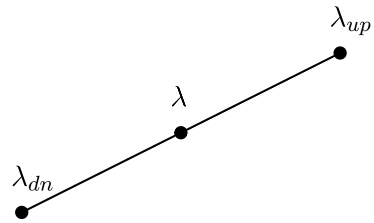


indicates that $H(\lambda) = H(\mu) > H(\nu)$.

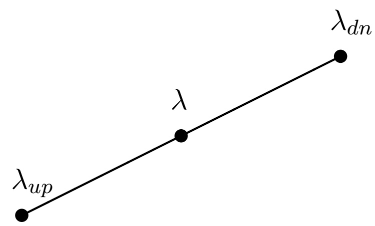
We claim that
$$|\{\lambda \in \mathscr{M}_r \,|\, d(\lambda) = 2,\ \min(H(\lambda_{dn}), H(\lambda_{up})) < H(\lambda)\}| \le 2|\cup_{i<r} \mathscr{M}_i|.$$

[1] We thank the referee for this useful suggestion.

To see this, let $\lambda \in \mathscr{M}_r$ such that $d(\lambda) = 2$ and $\min(H(\lambda_{dn}), H(\lambda_{up})) < H(\lambda)$. Then λ lies in a connected component C and a part of the graph of C is of either the form



or the form



Therefore, we always have either $\lambda_{dn} \in \bigcup_{i<r} \mathscr{M}_i$ or $\lambda_{up} \in \bigcup_{i<r} \mathscr{M}_i$. Since each vertex of Γ has at most two neighbors, the claim is proved.

Let

$$X := \{\lambda \in \mathscr{M}_r \,|\, d(\lambda) = 2,\ \min(H(\lambda_{dn}), H(\lambda_{up})) \geq H(\lambda)\}.$$

Then the above claim and Lemma 3.3 imply that

$$|X| \geq |\mathscr{M}_r| - 4|\cup_{i<r} \mathscr{M}_i|.$$

Now, to every $\lambda \in X$ we will associate a $\varphi(\lambda) := \mu \in \mathscr{P}$ such that $(\lambda, \mu) \in E$ and $\chi_\lambda(1)/4 < \chi_\mu(1) < \chi_\lambda(1)$. Let C be the connected component of Γ containing λ. If λ is the only vertex of $C \cap \mathscr{M}_r$ then $H(\lambda_{dn}) > H(\lambda)$ and $H(\lambda_{up}) > H(\lambda)$ and therefore by Lemma 3.4, we have

$$1 < \frac{H(\lambda_{dn})}{H(\lambda)}, \frac{H(\lambda_{up})}{H(\lambda)} < 4$$

so that both $\mu = \lambda_{dn}$ and $\mu = \lambda_{up}$ are good choices. Now we assume that $|C \cap \mathscr{M}_r| = 2$. Then, as observed above, the two vertices of $C \cap \mathscr{M}_r$ are either neighboring vertices in Γ or all the inner points of the subpath connecting them are elements from the set $\cup_{i<r}\mathscr{M}_i$. However, since $\min(H(\lambda_{dn}), H(\lambda_{up})) \geq H(\lambda)$, the two vertices of $C \cap \mathscr{M}_r$ must be neighboring vertices. We conclude that $|\{\lambda_{dn}, \lambda_{up}\} \cap \mathscr{M}_r| = 1$ and we just choose μ to be the vertex in $\{\lambda_{dn}, \lambda_{up}\}$ that is not in $\mathscr{M}_r$.

It remains to prove that the function φ we have just defined is injective. Note that $\varphi(\lambda)$ is always a neighbor of λ of higher level. On the other hand, by Lemma 3.4, every partition with two neighbors has at least one neighbor of higher level. We deduce that φ is injective and the proof is complete. $\square$

4. Theorems 1.1 and 1.2

We now show that Proposition 3.5 implies Theorem 1.2 when the cardinality of $\mathscr{M}_1$ is large enough.

Corollary 4.1. *If $|\mathscr{M}_1| \geq 32$, then Theorem 1.2 holds.*

Proof. Let

$$\mathscr{T} := \left\{ \chi \in \mathrm{Irr}(\mathsf{S}_n) \,\middle|\, \frac{b(\mathsf{S}_n)}{4} < \chi(1) < b(\mathsf{S}_n) \right\}.$$

By Proposition 3.5 (1) we have $|\mathscr{T}| \geq |\mathscr{M}_1| \geq 32$. Thus,

$$\sum_{\substack{\chi \in \mathrm{Irr}(\mathsf{S}_n) \\ \chi(1) \neq b(\mathsf{S}_n)}} \chi(1)^2 \geq \sum_{\chi \in \mathscr{T}} \chi(1)^2 > |\mathscr{T}| \cdot \left(\frac{b(\mathsf{S}_n)}{4} \right)^2 \geq 2b(\mathsf{S}_n)^2,$$

as desired. □

The case where $|\mathscr{M}_1|$ is small is handled by a different technique. From now on, for characters χ_1, χ_2 of a group G we write $\langle \chi_1, \chi_2 \rangle_G$ or simply $\langle \chi_1, \chi_2 \rangle$ if G is clear to denote their inner product.

Proposition 4.2. *Let $\lambda \in \mathscr{M}_1$ and let $\chi := \chi_\lambda$. If $n \geq 56$ and $|\mathscr{M}_1| \leq 31$, then*

$$\sum_{\substack{\langle \varphi, (\chi_{S_{n-1}})^{\mathsf{S}_n} \rangle \neq 0 \\ \varphi(1) < b(\mathsf{S}_n)}} \varphi(1)^2 > 2b(\mathsf{S}_n)^2.$$

In particular, Theorem 1.2 holds in this case.

Proof. By the branching rule we have

$$(\chi_{\mathsf{S}_{n-1}})^{\mathsf{S}_n} = |R(\lambda)| \cdot \chi + \sum_{i \neq j} \chi_{\lambda_{i \to j}},$$

where we recall that $R(\lambda)$ is the set of nodes that can be removed from Y_λ to obtain another Young diagram of size $n-1$, and $\lambda_{i \to j}$ denotes, if it is defined, the partition obtained from λ by moving the last node from row i to the end of the row j.

We also recall that $|R(\lambda)| \leq \frac{-1+\sqrt{1+8n}}{2}$ and if μ is a partition of $n-1$ then $|A(\mu)| < \frac{1+\sqrt{8n-7}}{2}$. Therefore the sum on the right hand side has at most

$$\frac{-1+\sqrt{1+8n}}{2} \cdot \frac{1+\sqrt{8n-7}}{2} < 2n$$

characters. Furthermore, χ appears at most $\frac{-1+\sqrt{1+8n}}{2} < \sqrt{2n}$ times, while there are at most 30 other characters in this sum with degree $b(\mathsf{S}_n)$. Therefore, since $(\chi_{\mathsf{S}_{n-1}})^{\mathsf{S}_n}(1) = nb(\mathsf{S}_n)$,

$$\sum_{\substack{\langle \varphi, (\chi_{S_{n-1}})^{\mathsf{S}_n} \rangle \neq 0 \\ \varphi(1) < b(\mathsf{S}_n)}} \varphi(1) > (n - \sqrt{2n} - 30) b(\mathsf{S}_n).$$

Using the Cauchy-Schwarz inequality and the fact that there are less than $2n$ constituents (irrespective of degree) in $(\chi_{\mathsf{S}_{n-1}})^{\mathsf{S}_n}$, we deduce that

$$\sum_{\substack{\langle\varphi,(\chi_{S_{n-1}})^{\mathsf{S}_n}\rangle\neq 0\\ \varphi(1)<b(\mathsf{S}_n)}} \varphi(1)^2 > \left(\frac{1}{\sqrt{2n}}(n-\sqrt{2n}-30)\right)^2 \cdot b(\mathsf{S}_n)^2.$$

It remains to check that

$$\frac{1}{\sqrt{2n}}(n-\sqrt{2n}-30) \geq \sqrt{2}$$

but this is clear as $n \geq 56$. □

We are now ready to finish the proof of Theorem 1.2.

Proof of Theorem 1.2. In light of Corollary 4.1 and Proposition 4.2, we only need to prove the theorem for $7 \leq n \leq 55$. We have done that by computations in [GAP] and the codes are available upon request.

For each $n \leq 75$, a partition corresponding to a character of S_n of the largest degree is available in [McK]. Let Y be the Young diagram corresponding to this partition. We consider all possible Young diagrams obtained from Y by moving one node from one row to another. For all those Young diagrams the degrees of the corresponding irreducible characters can be determined. If the degree of such a character coincides with the largest character degree of S_n, then it is excluded. We finally check that the sum of the squares of the remaining degrees is greater than $2b(\mathsf{S}_n)^2$, as desired. □

We now move on to a proof of Theorem 1.1. First we handle the case where A_n and S_n have the same largest character degree.

Proposition 4.3. *If $b(\mathsf{A}_n) = b(\mathsf{S}_n)$ then Theorem 1.1 holds.*

Proof. Recall that the restriction of each irreducible character of S_n to A_n is either irreducible or a sum of two irreducible characters of equal degree. Note also that, in the case $b(\mathsf{A}_n) = b(\mathsf{S}_n)$, every $\psi \in \mathrm{Irr}(\mathsf{A}_n)$ with $\psi(1) = b(\mathsf{A}_n)$ must be the (irreducible) restriction of some $\chi \in \mathrm{Irr}(\mathsf{S}_n)$ with $\chi(1) = b(\mathsf{S}_n)$. Therefore,

$$\sum_{\substack{\psi\in\mathrm{Irr}(\mathsf{A}_n)\\ \psi(1)<b(\mathsf{A}_n)}} \psi(1)^2 \geq \frac{1}{2} \sum_{\substack{\chi\in\mathrm{Irr}(\mathsf{S}_n)\\ \chi(1)<b(\mathsf{S}_n)}} \chi(1)^2.$$

Using Theorem 1.2, we obtain

$$\sum_{\substack{\psi\in\mathrm{Irr}(\mathsf{A}_n)\\ \psi(1)<b(\mathsf{A}_n)}} \psi(1)^2 > b(\mathsf{S}_n)^2 = b(\mathsf{A}_n)^2,$$

as desired. □

The proof of Theorem 1.1 in the case $b(\mathsf{A}_n) < b(\mathsf{S}_n)$ turns out to be more complicated. We will explain this in the rest of this section.

Let λ be the partition corresponding to a character of the largest degree of S_n. Then λ is self-conjugate as $b(\mathsf{A}_n) < b(\mathsf{S}_n)$. Lemma 3.3 guarantees that $d(\lambda) = 2$ and it follows that λ_{up} and λ_{dn} are *not* self-conjugate. In particular, $\chi_{\lambda_{up}}(1)$ and $\chi_{\lambda_{dn}}(1)$ are both at most $b(\mathsf{A}_n)$. Using Lemma 3.4, we deduce that

$$\frac{b(\mathsf{S}_n)^2}{b(\mathsf{A}_n)^2} = \frac{\chi_\lambda(1)^2}{b(\mathsf{A}_n)^2} \le \frac{\chi_\lambda(1)^2}{\chi_{\lambda_{up}}(1)\chi_{\lambda_{dn}}(1)} < 4,$$

which in turns implies that $b(\mathsf{S}_n)/2 < b(\mathsf{A}_n)$. In summary, we have

$$\frac{b(\mathsf{S}_n)}{2} < b(\mathsf{A}_n) < b(\mathsf{S}_n).$$

If there are two irreducible characters of S_n of the largest degree, then the associated partitions are both self-conjugate and so there are four irreducible characters of A_n of degree $b(\mathsf{S}_n)/2$, and we are done. So from now on we assume that there is only one irreducible character of degree $b(\mathsf{S}_n)$ of S_n. In other words, $|\mathscr{M}_1| = 1$.

If there is $\mu \in \mathscr{P}$ such that $b(\mathsf{A}_n) < \chi_\mu(1) < b(\mathsf{S}_n)$ then clearly μ must be self-conjugate. In this case A_n has two irreducible characters (lying under χ_λ) of degree $b(\mathsf{S}_n)/2$ and two irreducible characters (lying under χ_μ) of degree at least $b(\mathsf{A}_n)/2$, and we are done again. So we assume furthermore that $b(\mathsf{A}_n)$ is the second largest character degree of S_n, that is, $b(\mathsf{A}_n) = b_2$.

Proposition 4.4. *Assume that there is precisely one irreducible character of S_n of degree $b(\mathsf{S}_n)$ and $b(\mathsf{A}_n)$ is the second largest character degree of S_n. If $|\mathscr{M}_2| \ge 20$, then Theorem 1.1 holds.*

Proof. Proposition 3.5 (2) and the hypothesis $|\mathscr{M}_2| \ge 20$ imply that

$$\left|\left\{\nu \in \mathscr{P} \;\middle|\; \frac{b(\mathsf{A}_n)}{4} < \chi_\nu(1) < b(\mathsf{A}_n)\right\}\right| \ge 16.$$

Thus the sum of the squares of the degrees of irreducible characters of A_n lying under these characters ν is at least

$$16 \cdot \frac{1}{2}\left(\frac{b(\mathsf{A}_n)}{4}\right)^2 = \frac{b(\mathsf{A}_n)^2}{2}.$$

On the other hand, the sum of the squares of the degrees of the two characters of A_n lying under χ_λ is $b(\mathsf{S}_n)^2/2$, which is larger than $b(\mathsf{A}_n)^2/2$. So we conclude that

$$\sum_{\substack{\psi \in \mathrm{Irr}(\mathsf{A}_n)\\ \psi(1) < b(\mathsf{A}_n)}} \psi(1)^2 > b(\mathsf{A}_n)^2,$$

as the theorem claimed. □

Proposition 4.5. *Assume that there is precisely one irreducible character of S_n of degree $b(\mathsf{S}_n)$ and $b(\mathsf{A}_n)$ is the second largest character degree of S_n. Let $\mu \in \mathscr{M}_2$ and let $\chi := \chi_\mu$. If $n \ge 43$ and $|\mathscr{M}_2| \le 19$, then*

$$\sum_{\substack{\langle \varphi, (\chi_{\mathsf{S}_{n-1}})^{\mathsf{S}_n}\rangle \neq 0\\ \varphi(1) < b(\mathsf{A}_n)}} \varphi(1)^2 > 2b(\mathsf{A}_n)^2.$$

In particular, Theorem 1.1 holds in this case.

Proof. The proof goes along the same lines as that of Proposition 4.2 and so we will skip some details. First, by the branching rule,

$$(\chi_{\mathsf{S}_{n-1}})^{\mathsf{S}_n} = |R(\mu)| \cdot \chi + \sum_{i \neq j} \chi_{\mu_{i\to j}},$$

where $R(\mu)$ is the set of nodes that can be removed from Y_μ to obtain another Young diagram of size $n-1$, and $\mu_{i\to j}$ denotes, if it is defined, the partition obtained from μ by moving the last node from row i to the end of the row j.

In the sum on the right hand side, there are at most 18 irreducible characters (other than χ) with degree $b(\mathsf{A}_n)$, and at most one irreducible character with degree $b(\mathsf{S}_n)$. We recall that $|R(\mu)| \leq \frac{-1+\sqrt{1+8n}}{2} < \sqrt{2n}$ and $b(\mathsf{S}_n) < 2b(\mathsf{A}_n)$. Therefore,

$$\sum_{\substack{\langle \varphi, (\chi_{S_{n-1}})^{\mathsf{S}_n}\rangle \neq 0 \\ \varphi(1) < b(\mathsf{A}_n)}} \varphi(1) > (n - \sqrt{2n} - 20) b(\mathsf{A}_n).$$

As the sum on the left hand side has at most $2n$ terms, the Cauchy-Schwarz inequality then implies that

$$\sum_{\substack{\langle \varphi, (\chi_{S_{n-1}})^{\mathsf{S}_n}\rangle \neq 0 \\ \varphi(1) < b(\mathsf{A}_n)}} \varphi(1)^2 > \left(\frac{1}{\sqrt{2n}} (n - \sqrt{2n} - 20) \right)^2 \cdot b(\mathsf{A}_n)^2.$$

Now the inequality in the proposition follows as $\frac{1}{\sqrt{2n}}(n - \sqrt{2n} - 20) > \sqrt{2}$ when $n \geq 43$.

To see that Theorem 1.1 holds under the given hypothesis, we just observe that

$$\sum_{\substack{\psi \in \mathrm{Irr}(\mathsf{A}_n) \\ \psi(1) < b(\mathsf{A}_n)}} \psi(1)^2 > \frac{1}{2} \sum_{\substack{\langle \varphi, (\chi_{\mathsf{S}_{n-1}})^{\mathsf{S}_n}\rangle \neq 0 \\ \varphi(1) < b(\mathsf{A}_n)}} \varphi(1)^2 > b(\mathsf{A}_n)^2.$$

□

Finally we can prove Theorem 1.1 in the case $b(\mathsf{A}_n) < b(\mathsf{S}_n)$.

Proposition 4.6. *If $b(\mathsf{A}_n) < b(\mathsf{S}_n)$, then Theorem 1.1 holds.*

Proof. As discussed at the beginning of this section, it suffices to assume that there is precisely one irreducible character of S_n of degree $b(\mathsf{S}_n)$ and $b(\mathsf{A}_n)$ is the second largest character degree of S_n. Now the proposition follows from Propositions 4.4 and 4.5 when $n \geq 43$.

Let us now describe how we verify the theorem for $n < 43$. As pointed out earlier the partition λ corresponding to the largest degree in [McK] is self-conjugate. Denote the Young diagram corresponding to this partition by Y, so Y is symmetric. Then as before we consider all Young diagrams obtained from Y by moving one node from one row to another. Note that all these Young diagrams are not symmetric anymore and Y_{up} and Y_{dn} (the Young diagrams of λ_{up} and λ_{dn}) are among these diagrams. For such a Young diagram we compute by [GAP] the associated character degree. There are two cases:

1) $b(\mathsf{A}_n)$ is neither $\chi_{\lambda_{up}}(1)$ nor $\chi_{\lambda_{dn}}(1)$. We have

$$\chi_{\lambda_{up}}(1)^2 + \chi_{\lambda_{dn}}(1)^2 \geq 2\chi_{\lambda_{up}}(1)\chi_{\lambda_{dn}}(1) > \frac{\chi_\lambda(1)^2}{2} = \frac{b(\mathsf{S}_n)^2}{2}$$

where the inequality in the middle comes from Lemma 3.4. Using the two irreducible characters of A_n lying under χ_λ, we obtain the desired inequality.

2) $b(\mathsf{A}_n)$ is either $\chi_{\lambda_{up}}(1)$ or $\chi_{\lambda_{dn}}(1)$. In particular, the largest degree (among the degrees we have computed) falls into either Y_{up} or Y_{dn}. Then we just check that the sum of the squares of all other smaller degrees is bigger than the square of this largest degree. □

Theorem 1.1 is now a consequence of Propositions 4.3 and 4.6.

5. Theorem 1.3

In this section, we will prove Theorem 1.3, which is restated below for the reader's convenience. As the main ideas are basically the same as those in Sections 4, we will skip most of the details. Recall that

$$\varepsilon(G) := \frac{\sum_{\chi\in\mathrm{Irr}(G),\, \chi(1)<b(G)} \chi(1)^2}{b(G)^2}.$$

Theorem 5.1. *We have $\varepsilon(\mathsf{A}_n) \to \infty$ and $\varepsilon(\mathsf{S}_n) \to \infty$ as $n \to \infty$.*

Proof. Following the proofs of Corollary 4.1 and Proposition 4.2, we obtain

$$\sum_{\substack{\chi\in\mathrm{Irr}(\mathsf{S}_n)\\ \chi(1)<b(\mathsf{S}_n)}} \chi(1)^2 \geq \max\left\{\frac{|\mathscr{M}_1|}{16}, \frac{(n-\sqrt{2n}-(|\mathscr{M}_1|-1))^2}{2n}\right\} b(\mathsf{S}_n)^2,$$

which implies that

$$\varepsilon(\mathsf{S}_n) \geq \max\left\{\frac{|\mathscr{M}_1|}{16}, \frac{(n-\sqrt{2n}-(|\mathscr{M}_1|-1))^2}{2n}\right\}.$$

It now easily follows that $\varepsilon(\mathsf{S}_n) \to \infty$ as $n \to \infty$.

To estimate $\varepsilon(\mathsf{A}_n)$, we again consider two cases. If $b(\mathsf{A}_n) = b(\mathsf{S}_n)$, then we have

$$\varepsilon(\mathsf{A}_n) \geq \frac{1}{2}\varepsilon(\mathsf{S}_n)$$

and therefore there is nothing more to prove.

So from now on we assume that $b(\mathsf{A}_n) < b(\mathsf{S}_n)$. Let x be the number of irreducible characters of S_n of degree bigger than $b(\mathsf{A}_n)$. These characters produce $2x$ irreducible characters of A_n of degree at least $b(\mathsf{A}_n)/2$ (and less than $b(\mathsf{A}_n)$) and therefore

$$\sum_{\substack{\chi\in\mathrm{Irr}(\mathsf{A}_n)\\ \chi(1)<b(\mathsf{A}_n)}} \chi(1)^2 \geq 2x \cdot \left(\frac{b(\mathsf{A}_n)}{2}\right)^2,$$

which yields

$$\varepsilon(\mathsf{A}_n) \geq \frac{x}{2}. \tag{1}$$

Let y be the multiplicity of the character degree $b(\mathsf{A}_n)$ of S_n. Then, by Proposition 3.5 with $b_r = b(\mathsf{A}_n)$, we have

$$\left|\left\{\nu \in \mathscr{P} \,\middle|\, \frac{b(\mathsf{A}_n)}{4} < \chi_\nu(1) < b(\mathsf{A}_n)\right\}\right| \geq y - 4x.$$

Each ν in this set produces either one irreducible character of A_n of degree greater than $b(\mathsf{A}_n)/4$ or two irreducible characters of A_n of degree greater than $b(\mathsf{A}_n)/8$. Thus

$$\varepsilon(\mathsf{A}_n) \geq \frac{y-4x}{32}. \tag{2}$$

On the other hand, by following similar arguments as in the proof of Proposition 4.5, we get

$$\varepsilon(\mathsf{A}_n) \geq \frac{(n-\sqrt{2n}-2x-(y-1))^2}{2n}. \tag{3}$$

Now combining Inequalities 1, 2, and 3, we have

$$\varepsilon(\mathsf{A}_n) \geq \max\left\{\frac{x}{2}, \frac{y-4x}{32}, \frac{(n-\sqrt{2n}-2x-(y-1))^2}{2n}\right\}.$$

From this it is clear that $\varepsilon(\mathsf{A}_n) \to \infty$ as $n \to \infty$ and the proof is complete. □

Acknowledgement

The authors are grateful to the anonymous referee for insightful comments and suggestions that have significantly improved the exposition of the paper. They also thank Attila Maróti for several helpful comments.

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