BOUNDS IN GROUPS WITH TRIVIAL FRATTINI SUBGROUP

Z. HALASI and K. PODOSKI

Abstract

Let G be an arbitrary group with finite commutator subgroup G'. We prove that if the Frattini subgroup $\Phi(G)$ is trivial then $|G : Z(G)| \leq |G'|^2$, and this estimate is sharp. A special case of this result was conjectured by M. Herzog, G. Kaplan and A. Lev.

1 Introduction

Let G be an arbitrary group. Intuitively, both $|G : Z(G)|$ and $|G'|$ measure how abelian the group is. Therefore, it is important to find connections between these measures. The first result in this direction was Schur's classical theorem, which states that if $G/Z(G)$ is finite, then G' is finite as well. A simple method based on ultra products shows that there exists a function f such that if $|G : Z(G)| = n$ then $|G'| \leq f(n)$. The best bound was given by Wiegold [11] showing that

$$|G'| \leq n^{\frac{1}{2}\log_2 n}.$$

On the other hand, if $|G'| < \infty$, then $G/Z(G)$ is not necessary finite (infinite extraspecial p-groups provide examples). However, P. Hall [10, page 423] proved that if G' is finite then $G/Z_2(G)$ is also finite (where $Z_2(G)$ denotes the second member of the upper central series of G). It was proved in [9] that if $|G'| = k$ then

$$|G : Z_2(G)| \leq k^{c \log_2 k}$$

with an appropriate constant c.

Suppose that $Z_2(G) = Z(G)$. Then, of course, $|G : Z(G)| \leq k^{c\log_2 k}$. As an example in [9] shows there is no better bound for the index of the centre apart from the value of the constant c.

However, if we assume a stronger condition, namely, the Frattini subgroup $\Phi(G)$ is trivial (in this case $Z_2(G) = Z(G)$ automatically holds) then there exist better upper bounds. Using a result of I. M. Isaacs [8] for finite solvable groups, M. Herzog, G. Kaplan and A. Lev [6] proved that if $\Phi(G) = Z(G) = 1$, then

$$|G| \leq |G'|^3, \quad \text{even} \quad |G| \leq |G'|^{\frac{5}{2}} \quad \text{if} \quad |G| \text{ is odd.}$$

Furthermore, they conjectured that $|G| \leq |G'|^2$ for such groups. We prove this conjecture and extend it by showing the following.

Theorem 1.1. *Let G be a group such that G' is finite and $\Phi(G) = 1$. Then*

$$|G : Z(G)| \leq |G'|^2$$

Equality holds if and only if G is Abelian.

Remarks.

1. We note that in case of the Fitting subgroup $F(G)$ is trivial, the above result is a special case of [5, Theorem 10].

2. Let G be a Frobenius group such that $G \simeq Z_p \rtimes Z_{p-1}$ (p prime). Here $G' \simeq Z_p$ and $Z(G) = \Phi(G) = 1$. This shows that the above inequality is sharp.

3. If $\Phi(G) = 1$, there is a simple lower bound for $|G : Z(G)|$ in terms of $|G'|$. Since $G' \cap Z(G) = 1$ (see the proof of Lemma 2.1) we have that

$$|G'| \leq |G : Z(G)|.$$

Equality holds if and only if $G \simeq A \times B$ where A is an Abelian and B is a perfect group.

For infinite groups we prove the following

Theorem 1.2. *Let G be an infinite group such that $|G'| = \kappa$ (κ is an infinite cardinal) and $\Phi(G) = 1$. Then $|G : Z(G)| \leq 2^\kappa$ and this estimate is sharp.*

2 Proofs

First, we need the following simple lemma.

Lemma 2.1. *Let G be a group such that G' is finite and $\Phi(G) = 1$. Let D be the intersection of all non-normal maximal subgroups of G. Then $G/Z(G)$ is finite, $(G/Z(G))' \simeq G'$ and $Z(G/Z(G)) = \Phi(G/Z(G)) = 1$. Moreover, $D = Z(G)$.*

Proof. First, we note that every maximal subgroup of G contains either G' or $Z(G)$. Therefore $G' \cap Z(G) \leq \Phi(G) = 1$. This implies that $(G/Z(G))' \simeq G'$ and $Z_2(G) = Z(G)$, so $Z(G/Z(G)) = 1$. It is clear that D is a normal subgroup and every non-normal maximal subgroup contains $Z(G)$, thus $Z(G) \leq D$. Let M be a normal maximal subgroup. Since G/M is a cyclic group of order p we have that $G' \leq M$. This means that $D \cap G' \leq \Phi(G) = 1$ so $D \leq Z(G)$. Hence $D = Z(G)$ and $\Phi(G/Z(G)) = 1$. Finally, by P. Hall's Theorem [10] we obtain that $G/Z_2(G) = G/Z(G)$ is finite. □

By the previous lemma in order to prove Theorem 1.1 it is enough to show

Theorem 2.2. *Let $G \neq 1$ be a finite group such that $Z(G) = \Phi(G) = 1$. Then $|G| < |G'|^2$.*

We shall need the following theorem

Theorem 2.3. (M. Aschbacher, R. M. Guralnick [2]) *If G is a primitive permutation group of degree n, then either G has prime order n or $|G : G'| < n$.*

Since G' is transitive so $|G'| \geq n$ in the preceding theorem we have

Corollary 2.4. *If G is a primitive permutation group of degree n, then either G has prime order n or $|G| < |G'|^2$.*

Before we proceed to see how an induction argument gives the general result we mention an easy fact.

Lemma 2.5. *If N is normal in the finite group G, then $\Phi(N) \leq \Phi(G)$.*

Proof. Let M be a maximal subgroup not containing $\Phi(N)$. Thus, $G = M\Phi(N) = MN$ and so in particular, $N = (M \cap N)\Phi(N)$, when a $N = M \cap N$, a contradiction. □

We now prove the main result

Proof of Theorem 2.2. Let $M_1, \ldots, M_t$ be non-normal maximal subgroups of G with core K_i (i.e. K_i is the intersection of the all conjugates of M_i). Choose t minimal such that $\cap_{i=1}^{t} K_i = 1$ (the intersection of all non-normal maximal subgroups of G is trivial by Lemma 2.1).

We induct on $|G|$ and take t to be minimal. Note also the result holds for $t = 1$ by Corollary 2.4. So $t > 1$.

Let $N = K_1 \cap \cdots \cap K_{t-1}$. Then $N \neq 1$ by the minimality of t. Also, N is embedded in the primitive permutation group G/K_t.

We claim that $|N| < |G' \cap N|^2$. Suppose that $Z(N) \neq 1$. Then G/K_t is a primitive permutation group containing a normal abelian subgroup. This means that $G/K_t \simeq Z(N) \rtimes S$, where S is a stabilizer subgroup of G/K_t. Thus $N \leq C_{G/K_t}(Z(N)) = Z(N)$ so $N = Z(N)$ is the socle of G/K_t and in particular, N is abelian and is a minimal normal subgroup of G. So $N \leq G'$ and the claim follows in that case. Otherwise, $Z(N) = 1 = \Phi(N)$ and by induction $|N| < |N'|^2$, whence the claim.

Since N is the intersection of some non-normal maximal subgroups, we have $\Phi(G/N) = Z(G/N) = 1$. By induction, the theorem holds for G/N. Thus,

$$|G| < |N||(G/N)'|^2 = |N||G'N : N|^2 = |N||G' : G' \cap N|^2 = |G'|^2 \frac{|N|}{|N \cap G'|^2} < |G'|^2.$$

□

Proof of Theorem 1.2. It is proved in [9] that $|G : Z_2(G)| \leq 2^{\kappa}$. As in Lemma 2.1 we get $Z_2(G) = Z(G)$. An example in [9] shows the sharpness of the inequality. □

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Zoltán Halasi
Central European University
Department of Mathematics
and Its Applications
H-1051 Budapest
Hungary
e-mail: haca@cs.elte.hu

Károly Podoski
Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
P. O. Box 127
H-1364 Budapest
Hungary
e-mail: pcharles@cs.elte.hu