On the characters of the unit group of DN-algebras

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Abstract

In this paper we examine the structure and the characters of the unit group of some special algebras, called DN-algebras and we prove that the irreducible characters are induced from linear characters of the unit group of some DN-subalgebras. This extends a result of B. Szegedy.

1 Introduction

Let K be a finite field with q elements and of characteristic p. Then a Sylow p-subgroup of $GL(n, K)$ consists of all unipotent upper triangular matrices in $GL(n, K)$, and its normalizer, called the Borel subgroup of $GL(n, K)$, is the group of all invertible upper triangular matrices in $GL(n, K)$. It was proved by B. Szegedy in [4] that these groups are M-groups, i.e., all irreducible complex characters can be obtained by induction from linear characters of subgroups. (He was motivated by a question of M. Geck.) In fact B. Szegedy has proved a more general statement. He defined the class of DN-algebras in the following way:

Definition 1.1. (Szegedy [4]) *Let A be a finite dimensional algebra with unity over the finite field K. We say that A is a DN-algebra if the set of the nilpotent elements is an ideal of A (this ideal is equal to the Jacobson radical of A, denoted by $J(A)$), and $A/J(A)$ is isomorphic to a direct sum of copies of K.*

Note that the algebra of $n \times n$ upper triangular matrices over K is a DN-algebra and its unit group is equal to the Borel subgroup of $GL(n, K)$. So the following theorem shows that the Borel subgroup is an M-group.

Theorem 1.2. (Szegedy [4]) *If A is a DN-algebra over the q-element field, then the unit group of A (denoted by $U(A)$) is an M-group. Moreover, in $U(A)$ the sizes of conjugacy classes and the degrees of irreducible characters have the form $q^r(q-1)^s$ for some $r \geq 0$, $s \geq 0$.*

The unique Sylow p-subgroup of $U(A)$ has the form $1+J(A)$. Such subgroups are called algebra groups by Isaacs [3]. By Theorem 1.2 in [1] the irreducible characters of $1 + J(A)$ are induced from linear characters of algebra subgroups, that is, subgroups of the form $1 + U$ where U is some multiplicatively closed subspace of $J(A)$. Comparing this result with the result of B. Szegedy it is natural to ask whether the same holds for the unit group of a DN-algebra. It is true indeed, and our goal is to prove the following theorem:

Theorem 1.3. *Let A be a DN-algebra and let $U(A)$ denote the unit group of A. Then for every $\omega \in Irr(U(A))$ there exist a subalgebra $C \leq A$ and a linear character λ of $U(C)$ such that $\omega = \lambda^{U(A)}$.*

Remarks:

- The subalgebras of a DN-algebra containing 1 are DN-algebras themselves by a lemma of B. Szegedy [4, Lemma 2.2].
- It is easy to check that the number of units in a DN-algebra over the field $\mathbb{F}_q$ has the form $q^r(q-1)^s$ where r and s are nonnegative integers. Hence our Theorem 1.3 implies that the degrees of irreducible characters of $U(A)$ have the same form.
- If A is a DN-algebra over the field $K \simeq \mathbb{F}_2$, then $U(A) = 1 + J(A)$, so in this case Theorem 1.3 says the same as Theorem 1.2 in [1]. Hence we assume $K \not\simeq \mathbb{F}_2$ in the following.
- In the case when $A/J(A) \simeq K$, i.e., if A is a local algebra, $U(A)$ can be written in the form $U(A) = (K^* \cdot 1) \times (1 + J(A))$. So, if $\omega \in \mathrm{Irr}(U(A))$, then $\omega = \mu \times \chi$ where $\mu \in \mathrm{Irr}(K^* \cdot 1)$ and $\chi \in \mathrm{Irr}(1 + J(A))$. If $C \leq J(A)$ is a multiplicatively closed subspace and $\lambda \in \mathrm{Irr}(1+C)$ such that $\lambda^{1+J(A)} = \chi$, then $K \cdot 1 + C$ is a subalgebra of A and $\mu \times \lambda \in \mathrm{Irr}(U(K \cdot 1 + C))$ is a linear character. Furthermore, $(\mu \times \lambda)^{U(A)} = \mu \times \chi = \omega$.

2 The structure of DN-algebras

In this section we prove some lemmas about the structure of DN-algebras which will be used in the next section.

Lemma 2.1. *Let A be a DN-algebra over the finite field K and let $J = J(A)$ denote the Jacobson radical of A.*

(a) There is a set of non-zero orthogonal idempotents $e_1, e_2, \ldots, e_k \in A$, where $e_1 + e_2 + \ldots + e_k = 1$ and k is the dimension of A/J over K.

(b) Let $B = Ke_1 \oplus Ke_2 \oplus \cdots \oplus Ke_k$. Then $A = B + J$ and $B \cap J = 0$. Hence $U(A) = (1 + J) \rtimes U(B)$ is a semidirect product.

(c) If M is a B-bimodule, then it is the direct sum of the homogeneous sub-bimodules e_iMe_j.

(d) If $M_1 \leq M$ are B-bimodules, then there exists a sub-bimodule $M_2 \leq M$, such that $M = M_1 \oplus M_2$.

(e) Every B-bimodule is a direct sum of one-dimensional sub-bimodules.

(f) If $u \in M$ generates a one-dimensional sub-bimodule, then there exist uniquely determined idempotents $e_l(u), e_r(u) \in \{e_1, e_2, \ldots, e_k\}$ such that $e_l(u)u = ue_r(u) = u$.

(g) If $u, v \in J$ both generate one-dimensional B-bimodules and $e_l(u) \neq e_r(v)$, then $vu = 0$.

Proof. Part (a) follows by "lifting idempotents" (see [6, Corollary 1.7.4]).
To prove (b) it is clear that B is a semisimple algebra and J is a nilpotent ideal of A such that $\dim B = \dim A/J$, so $B \cap J = 0$ and $A = B + J$. Now, let $\varphi : A \to A/J$ be the natural algebra homomorphism. Then the restriction of φ to $U(A)$ will be a surjective group homomorphism $U(A) \to U(A/J(A)) \simeq U(B)$ with kernel $1 + J$, and $U(A) = (1 + J) \rtimes U(B)$ follows.
Let $\sum_m \alpha_m e_m, \sum_m \beta_m e_m \in B$ and $e_i v e_j \in e_i M e_j$. Then

$$\sum_m \alpha_m e_m (e_i v e_j) \sum_m \beta_m e_m = (\alpha_i \beta_j) e_i v e_j,$$

which proves the $e_i M e_j$-'s are homogenous submodules for all $1 \le i, j \le k$. If $v \in M$, then there is a unique decomposition of $v = \sum v_{ij}$ such that $v_{ij} \in e_i M e_j$, namely $v_{ij} = e_i v e_j$. Hence M is the direct sum of the $e_i M e_j$-'s and (c) is proved.
If $M_1 \le M$ are B-bimodules, then for any $1 \le i, j \le k$ we have $e_i M_1 e_j \le e_i M e_j$. Using that $e_i M e_j$ is homogenous it follows that all of its subspaces are sub-bimodules, so clearly there exists a sub-bimodule $M_{2,ij} \le e_i M e_j$ such that $e_i M e_j = e_i M_1 e_j \oplus M_{2,ij}$. Now, let $M_2 = \oplus M_{2,ij}$. It is clear that $M = M_1 \oplus M_2$.
(e) follows easily from (c), because any subspace of a homogenous submodule is also a submodule, so any direct decomposition of a homogenous submodule to one dimensional subspaces is a direct decomposition to sub-bimodules.
If Ku is a B-bimodule, then using (c) there exist uniquely determined idempotents $e_l(u), e_r(u)$ such that $e_l(u) K u e_r(u) = Ku$. Then $e_i u = 0$ for all $e_i \ne e_l(u)$ and $u e_j = 0$ for all $e_j \ne e_r(u)$. Finally, $e_l(u) = (\sum e_i) u = u = u(\sum e_j) = u e_r(u)$.
Finally, (g) follows from the identity $vu = v e_r(v) e_l(u) u = 0$. □

In the following we fix a set of orthogonal idempotents $\{e_1, e_2, \ldots, e_k\} \subseteq A$ and the subalgebra $B = Ke_1 \oplus Ke_2 \oplus \cdots \oplus Ke_k$. So $U(A) = (1 + J) \rtimes U(B)$.

Lemma 2.2. *Let $V \le J$ be a B-bimodule with a direct decomposition $V = Kx_1 \oplus Kx_2 \oplus \cdots \oplus Kx_m$ to one-dimensional sub-bimodules, and let $0 \ne W < V$ be a subspace satisfying the following conditions:*

1. *$U(B)$ normalizes W.*
2. *$V' \cap W = 0$ for all proper sub-bimodules $V' < V$.*

Then

(a) $W \not\le \oplus_{j \ne i} Kx_j$ for $1 \le i \le m$.

(b) For $1 \le i \ne j \le m$ we have $e_l(x_i) \ne e_l(x_j)$ and $e_r(x_i) \ne e_r(x_j)$.

Proof. Let $V_i = \oplus_{j \ne i} Kx_j$. Then $V_i < V$ is a proper sub-bimodule, so $V_i \cap W = 0$ by condition 2 and (a) follows.

If $e_l(x_i) = e_l(x_j)$ for $i \ne j$ then Kx_i and Kx_j are isomorphic as left submodules. So each subspace of $Kx_i \oplus Kx_j$ is a left B-submodule. Then $V' = \oplus_{s \notin \{i,j\}} Kx_s \oplus W$ is both left submodule and invariant under the action of $U(B)$ by conjugation. Hence if $v \in V'$ and $b \in U(B)$ then $vb = b(b^{-1} v b) \in V'$. Furthermore, the subspace generated by $U(B)$ is B. So V' is a proper sub-bimodule containing W, contrary to the second assumption. The proof of the statement $e_r(x_i) \ne e_r(x_j)$ for $i \ne j$ is similar. □

We say that $1+I \leq 1+J$ is an ideal subgroup, if $I \leq J$ is an ideal of A. We note that in this case $1+I$ is a normal subgroup of $U(A)$. (Isaacs defined the ideal subgroup in [3] in a different way: He called $1+I \leq 1+J$ an ideal subgroup, if I is an ideal of J.)

Lemma 2.3. *Let $1+I \leq 1+J$ be an ideal subgroup. Then $1+I$ is generated as a group by the set $Y = \{1+x \,|\, x \in I,\ Kx \text{ is a } B\text{-sub-bimodule}\}$.*

Proof. We prove by reverse induction that $Y_k = Y \cap (1+J^k)$ generates $1+I_k = 1+(I \cap J^k)$ for all k. This is clear if $J^k = 0$. Assuming that Y_k generates $1+I_k$ for some k we can choose $x_1, x_2, \ldots, x_l \in I_{k-1}$ by Lemma 2.1 such that $I_{k-1} = I_k \oplus Kx_1 \oplus \cdots \oplus Kx_l$ and each Kx_i is a B-sub-bimodule. Then $\{1+Kx_i+I_k,\ i=1,2,\ldots,l\}$ generates $1+(I_{k-1}/I_k) \simeq (1+I_{k-1})/(1+I_k)$, because $(1+x+I_k)(1+y+I_k) = 1+x+y+I_k$ for $x, y \in I_{k-1}$. So $\{1+Kx_i \,|\, i = 1,2,\ldots,l\} \cup Y_k \subseteq Y_{k-1}$ generates $1+I_{k-1}$. □

3 Characters of the unit group of a DN-algebra

We prove Theorem 1.3 in this section. We use all the notations of the previous section. The essential point of our proof is to prove that if χ is a non-linear, $U(B)$-invariant irreducible character of $1+J$ then it can be obtained by induction from a proper ideal subgroup $L \geq 1+J^2$. To see this we examine the action of $U(B)$ by conjugation on J as well as on $\mathrm{Irr}(1+J)$.

Let $\mathrm{Irr}_{U(B)}(L)$ denote the set of all $U(B)$-invariant irreducible characters for an ideal subgroup L. For a subspace $V \subseteq J$ and $\varphi \in \mathrm{Irr}(L)$ let $I_V(\varphi) = \{v \in V \,|\, 1+v \in I_{1+J}(\varphi)\}$.

Lemma 3.1. *Let $L \leq 1+J$ be an ideal subgroup. If $\varphi \in Irr(L)$, then the inertia subgroup $I_{U(B)}(\varphi)$ is the unit group of a subalgebra of B.*

Proof. If $L = 1+I$, then I is an ideal of A, so $B+I$ is a subalgebra of A with unit group $U(B+I) = L \rtimes U(B)$. Hence we can assume without loss of generality that $L = 1+J$. $U(B)$ acts on $1+J$ by conjugation. This action determines an action of $U(B)$ on $\mathrm{Irr}(1+J)$ and an action on $\mathrm{Cl}(1+J)$, on the set of conjugacy classes of the group $1+J$. Here $(|U(B)|, |1+J|) = 1$ and $U(B)$ is solvable (even abelian), so by Theorem 13.24 in [2] the two actions are permutation isomorphic. Hence $I_{U(B)}(\varphi) = St(\mathcal{C})$ for some $\mathcal{C} \in \mathrm{Cl}(1+J)$, where $St(\mathcal{C})$ denotes the stabilizer of $\mathcal{C}$ in $U(B)$. If $x \in \mathcal{C}$ is an arbitrary element, then $St(\mathcal{C}) = U(B) \cap (C_{U(A)}(x)(1+J))$. Here $C_{U(A)}(x) = U(C_A(x))$ is the unit group of the subalgebra $C_A(x)$. Hence $I_{U(B)}(\varphi) = U(B \cap (C_A(x)+J))$. Furthermore, $B \cap (C_A(x)+J)$ is a subalgebra of B and the proof is complete. □

The following consequences of Glaubermann's Lemma will be used to find $U(B)$-invariant characters of ideal subgroups.

Lemma 3.2. *Let S act on G such that $(|S|, |G|) = 1$ and let $N < L < G$ be S-invariant normal subgroups of G.*

1. *Let $\chi \in Irr(G)$ be S-invariant. Then χ_N has an S-invariant irreducible constituent. Furthermore, any two such constituents are conjugate via an element of $C_G(S)$.*

2. *Let $\chi \in Irr(G)$ and $\varphi \in Irr(N)$ be S-invariant such that φ is a constituent of χ_N. Then there is an S-invariant $\psi \in Irr(L)$ between χ and φ, that is, ψ is a constituent of χ_L and φ is a constituent of ψ_N.*

Proof. The first part of 1 is a special case of [2, Theorem 13.27]. This Theorem also says that the hypotheses of Glaubermann's Lemma are satisfied, so the second part of 1 follows from [2, Corollary 13.9]

To see 2 we can apply 1 twice. So we get S-invariant characters $\psi' \in \mathrm{Irr}(L)$ and $\varphi' \in \mathrm{Irr}(N)$ such that ψ' is a constituent of χ_L and φ' is a constituent of ψ'_N, so it is also a constituent of χ_N. By the second part of 1 there is a $c \in C_G(S)$ such that $\varphi = (\varphi')^c$. Then $\psi = (\psi')^c \in \mathrm{Irr}(L)$ is an S-invariant character between χ and φ. □

Lemma 3.3. *Suppose that $1+J$ is a K-algebra group, where K is a finite field with q elements and of characteristic p. Let $J' \leq J$ be an ideal of J and let $\lambda \in Irr(1+J')$ be a linear character of $1+J'$ such that $1+J^2 \leq I_{1+J}(\lambda)$. Then $I_J(\lambda)$ is an ideal of J over K.*

Proof. It is clear that $I_J(\lambda)$ is an ideal of J over $\mathbb{F}_p$. So it is enough to show that if $x \in I_J(\lambda)$, then $Kx \subseteq I_J(\lambda)$.

By the definition of Isaacs [3] a subgroup $S \leq 1+J$ is called strong, if $|S \cap H|$ is a power of q for all algebra subgroups $H \leq 1+J$. We know that $I_{1+J}(\lambda)$ is a strong subgroup by [3, Theorem 8.3]. Let $x \in J$ be an arbitrary element. Then $1+Kx+J^2$ is an algebra subgroup, so either $1+Kx+J^2 \leq I_{1+J}(\lambda)$ or $(1+Kx+J^2) \cap I_{1+J}(\lambda) = 1+J^2$. The result follows. □

The following lemma will be used in two particular cases: If $L \geq 1+J^2$ or if $L = 1+J^k$ for some k.

Lemma 3.4. *Let $L \leq 1+J$ be an ideal subgroup of $U(A)$. Assume that $\lambda \in Irr_{U(B)}(L)$ is a linear character such that $J^2 \leq I_J(\lambda)$. Then $I_J(\lambda)$ is an ideal of A.*

Proof. Let $I' \geq J^2$ be the unique maximal ideal in $I_J(\lambda)$. If $I' = J$, then there is nothing to prove. Otherwise, let us choose a B-sub-bimodule $V \leq J$ such that $J = I' \oplus V$ by Lemma 2.1 (d). Let $W = I_V(\lambda)$, so $I_J(\lambda) = I' + W$. What we need to show is that $W = 0$. Assume by contradiction that $0 \neq W < V$.

If J' is a proper ideal of A such that $J > J' \geq I'$, then $J'^2 \leq J^2 \cap J' \leq I_{J'}(\lambda)$, so $I_{J'}(\lambda)$ is an ideal of A by inductive hypothesis. It follows that $I_{J'}(\lambda) = I'$ for such ideals. If $V' < V$ is a proper submodule then $I' + V' < J$ is a proper ideal of A, hence $V' \cap W = I_{V'}(\lambda) = 0$. Furthermore, W is a K-subspace by Lemma 3.3 and $U(B)$ normalizes W. Hence all the assumptions of Lemma 2.2 hold for V and W. Let $V = Kx_1 \oplus Kx_2 \oplus \ldots \oplus Kx_l$ be a direct decomposition of V to one dimensional B-sub-bimodules and let $Kz \leq I$ be a B-sub-bimodule. Choosing an $x \in \{x_1, x_2, \ldots, x_l\}$ we will prove that $[1+x, 1+z] \subseteq \ker\lambda$, or equivalently $\lambda^{1+x}(1+z) = \lambda(1+z)$. We distinguish two cases.

If $e_l(z) = e_r(x)$ and $e_r(z) = e_l(x)$, then it follows directly from Lemma 2.2 (b) and from Lemma 2.1 (g) that $x_j z = z x_j = 0$ for all $x_j \neq x$. Hence $[1+Kx_j, 1+z] = 0$ for all $x_j \neq x$. On the other hand $[1+I'+W, 1+z] \subseteq \ker\lambda$ by the definition of I' and W. Clearly $\{1+t \in 1+J \mid \lambda^{1+t}(1+z) = \lambda(1+z)\}$ is a subgroup of $1+J$, so it is enough to prove that

$$\langle 1+Kx_j \,|\, x_j \neq x \rangle \, (1+I'+W) = 1+J.$$

It is clear that $1 + J/1 + I'$ is isomorphic to the additive group of V. On the other hand V is generated (as an additive group) by the set $\cup_{x_j \neq x} K x_j \cup W$ by Lemma 2.2 (a). Therefore, $\langle 1 + K x_j \,|\, x_j \neq x \rangle\, (1 + I' + W)/(1 + I') \simeq V$, which proves the above identity. Hence $\lambda^{1+x}(1 + z) = \lambda(1 + z)$, as we have claimed.

Now assume that, for example, $e_l(z) \neq e_r(x)$. Then $xz = 0$. Easy calculation shows that

$$[1 + x, 1 + z] = 1 + (1 + x)^{-1}(1 + z)^{-1}(xz - zx) = (1 - zx + z^2x - z^3x + \ldots).$$

It follows that $[1 + x, 1 + z] - 1 \in zAx$, so $([1 + x, 1 + z] - 1)^2 = 0$.

Let $a \in K^*$ be a non-zero field element. Then there exists $b \in U(B)$ such that $b^{-1}z = az$ and $xb = x$. Conjugating the commutator $[1 + x, 1 + z]$ by b we get

$$[1 + x, 1 + z]^b = (1 - zx + z^2x - z^3x + \ldots)^b = (1 - b^{-1}zxb + b^{-1}z^2xb - \ldots) = 1 + a(-zx + z^2x - z^3x + \ldots) = 1 + a\,([1 + x, 1 + z] - 1)\,.$$

Let $C = K\,([1 + x, 1 + z] - 1)$. We have already seen that $([1+x, 1+z]-1)^2 = 0$, hence C is a one dimensional algebra and $1 + C$ is an algebra subgroup. The above formula shows that all elements of $(1 + C) \setminus \{1\}$ are conjugate under the action of $U(B)$. Using the fact that λ is a $U(B)$-invariant character it follows that λ_{1+C} is a linear character such that λ_{1+C} is constant on $(1 + C) \setminus \{1\}$. Then $\lambda_{1+C} = 1_{1+C}$, i.e., $[1 + x, 1 + z] \in \ker \lambda$.

Let $Y = \{1 + z \,|\, Kz \leq I \text{ is a } B\text{-sub-bimodule}\}$. Then $\lambda^{1+x}(y) = \lambda(y)$ for all $x \in \{x_1, x_2, \ldots, x_l\}$ and for all $y \in Y$. But the subgroup generated by Y is equal to $1 + I$ by Lemma 2.3 and the values of a linear character on a set of generators determine the character. Hence $\{x_1, x_2, \ldots, x_l\} \subseteq I_V(\lambda)$. But $I_V(\lambda)$ is a K-subspace with basis $\{x_1, x_2, \ldots, x_l\}$, so $I_V(\lambda) = V$, which is a contradiction to our assumption $I_V(\lambda) = W < V$. □

The key step of the proof is the following theorem:

Theorem 3.5. *Suppose that A is a DN-algebra over the field K and $U(A) = (1+J) \rtimes U(B)$. Let $\chi \in Irr_{U(B)}(1+J)$ be a non-linear, $U(B)$-invariant character of $1 + J$. Then there exist a proper ideal subgroup $L < 1 + J$ and a character $\psi \in Irr_{U(B)}(L)$ such that $\psi^{1+J} = \chi$.*

Proof. Let $\vartheta \in \mathrm{Irr}_{U(B)}(1 + J^2)$ be a constituent of χ_{1+J^2}.

Assuming first that ϑ is a linear character, choose a maximal ideal subgroup L such that ϑ is extendible to L. Then $L < 1+J$, because χ is not a linear character. Furthermore, there exists a $\psi \in \mathrm{Irr}_{U(B)}(L)$ between χ and ϑ by Lemma 3.2 and this character is an extension of ϑ to L, because $L/(1 + J^2)$ is abelian. Then $I_{1+J}(\psi)$ is an ideal subgroup by Lemma 3.4. We prove that $I_{1+J}(\psi) = L$. Otherwise, we could choose an ideal subgroup L' such that $|L' : L| = q$ and $L' \leq I_{1+J}(\psi) \leq I_{1+J}(\vartheta)$. The degree of each irreducible character of an ideal subgroup is a power of q by a theorem of Isaacs [3, Theorem A], so either $\psi^{L'}$ is irreducible, or ψ is extendible to L'. However, $I_{L'}(\psi) = L' > L$ so $\psi^{L'}$ cannot be irreducible by Problem 6.1 in [2], and ψ cannot be extendible to L' by the maximal choice of L. So $I_{1+J}(\psi) = L$ and $\psi^{1+J} = \chi$.

If ϑ is not a linear character, then $I_{1+J}(\vartheta) < 1 + J$ by [1, Theorem 1.3]. We prove that there exists a proper ideal subgroup $L \geq I_{1+J}(\vartheta)$.

To see this let $k > 2$ be the smallest integer such that $1 + J^k \leq Z(\vartheta)$. Then $\vartheta_{1+J^k} = \vartheta(1) \cdot \lambda$, where $\lambda \in \mathrm{Irr}_{U(B)}(1 + J^k)$ is a linear character such that $I_{1+J}(\lambda) \geq I_{1+J}(\vartheta) \geq 1 + J^2$. Applying Lemma 3.4 we get that $I_{1+J}(\lambda)$ is an ideal subgroup of $1 + J$, so it remains to show that $I_{1+J}(\lambda) < 1 + J$. On the one hand, $[1 + J^2, 1 + J^{k-1}] \not\leq \ker\vartheta = \ker\lambda$ by the minimal choice of k. On the other hand, $[1 + J^2, 1 + J^{k-1}] \leq [1 + J, 1 + J^k]$ by [1, Theorem 1.4]. Hence $[1 + J, 1 + J^k] \not\leq \ker\lambda$, which is equivalent to the inequality $I_{1+J}(\lambda) < 1 + J$. So $L = I_{1+J}(\lambda)$ is a proper ideal subgroup containing $I_{1+J}(\vartheta)$.

By the Clifford correspondence [2, Theorem 6.11] there exists an irreducible character φ of $I_{1+J}(\vartheta)$ such that $\varphi^{1+J} = \chi$. So $\chi = (\varphi^L)^{1+J}$ and χ is induced from a character of the proper ideal subgroup L. It follows directly from the Clifford theorem [2, Theorem 6.2] that $\chi = \psi^{1+J}$ for each component ψ of χ_L. Finally, we can choose ψ such that $\psi \in \mathrm{Irr}_{U(B)}(L)$ by Lemma 3.2. The proof is complete. □

Proof of Theorem 1.3

By the transitive property of induction and by the fact that all subalgebras of a DN-algebra are again DN-algebras it is enough to prove that if $\omega \in \mathrm{Irr}(U(A))$ is not a linear character then there exists a proper subalgebra $A' < A$ such that ω is induced from a character ω' of $U(A')$.

Let $U(A) = (1 + J) \rtimes U(B)$ and let χ be a component of ω_{1+J}. Then $I_{U(B)}(\chi) = U(B')$ for a subalgebra B' of B by Lemma 3.1. Hence $I_{U(A)}(\chi) = U(A')$ is the unit group of the subalgebra $A' = B' + J$. By Theorem 6.11 in [2] ω is induced from a character ω' of $U(A')$. If $B' < B$ then $A' = B' + J$ is a proper subalgebra of A.

Assume that $\chi \in \mathrm{Irr}_{U(B)}(1+J)$. Then χ is extendible to $U(A)$ by Corollary 6.28 in [2]. (Note that $(|U(A) : 1 + J|, |1 + J|) = 1$.) Furthermore, ω is an extension of χ by Corollary 6.17 in [2]. So χ is not a linear character.

By Theorem 3.5 there exist a proper ideal subgroup $L = 1 + I < 1 + J$ and a character $\psi \in \mathrm{Irr}_{U(B)}(1 + I)$ such that $\psi^{1+J} = \chi$. Then $A' = B + I$ is a proper subalgebra of A. Let φ be an extension of ψ to $U(A')$ by Corollary 6.28 in [2]. Then $(\varphi^{U(A)})_{1+J} = \psi^{1+J} = \chi$ by [2, Problem 5.2], so $\varphi^{U(A)} = \omega\mu$ for some $\mu \in \mathrm{Irr}(U(A)/(1 + J))$. Let $\omega' = \varphi\mu^{-1}_{U(A')}$. Hence $(\omega')^{U(A)} = \omega$ using [2, Problem 5.3]. The proof is complete. □

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