On the characters and commutators of finite algebra groups

Zoltán Halasi

Abstract

Let A be a finite dimensional algebra over a finite field and let $J(A)$ be the Jacobson radical of A. We prove a new commutator identity in the group $G = 1+J(A)$ and using this identity we show that every irreducible character of G is induced from a linear character of some subgroup of the form $H = 1+U$, where U is a subalgebra of $J(A)$. This answers a question of I. M. Isaacs.

1 Introduction

Let A be a finite $\mathbb{F}_q$-algebra with identity, where $\mathbb{F}_q$ is a finite field of characteristic p, and let $J = J(A)$ be the Jacobson radical of A. Then the group $1 + J$ is called an **$\mathbb{F}_q$-algebra group**. The subgroups of an algebra group which are of the form $1+U$, where $U \leq J$ is a subalgebra of J are called **algebra subgroups**. If A is a finite dimensional algebra, then $J(A)$ is a nilpotent algebra, so $1+J(A)$ is a nilpotent subgroup of the group of units of A. If J is a finite dimensional nilpotent algebra then we can define an algebra $A = \mathbb{F}_q \cdot 1+J$ such that $J = J(A)$. So the algebra groups are exactly the groups $1+J$ associated to the finite nilpotent algebras J with the multiplication $(1 + j_1) \cdot (1 + j_2) := 1 + j_1 + j_2 + j_1 j_2$. The following theorem was proved by I. M. Isaacs [2, Theorem A].

Theorem 1.1. (Isaacs [2]) *Let G be an $\mathbb{F}_q$-algebra group. Then all irreducible complex characters of G have q-power degree.*

In this paper we prove the following:

Theorem 1.2. *Let G be an $\mathbb{F}_q$-algebra group and $\chi \in \mathrm{Irr}(G)$. Then there exist an $\mathbb{F}_q$-algebra subgroup $H \leq G$ and a linear character λ of H such that $\chi = \lambda^G$.*

This result appears as a question in the paper of Isaacs and it was proved by Carlos A. M. André in [3] for the case $J^p = 0$, where $p = \mathrm{char}\, \mathbb{F}_q$. We note that Theorem A, Corollary B and Theorem C in [2] are immediate consequences of our Theorem 1.2. Our proof will be based on the following observation:

Theorem 1.3. *Let $G = 1 + J$ be an $\mathbb{F}_q$-algebra group and $\varphi \in \mathrm{Irr}(1 + J^2)$. If φ is a G-invariant character, then φ is linear.*

For a positive integer k, $1+J^k$ is an algebra subgroup of $G = 1+J$. We have found the following connection between the commutators of these subgroups:

Theorem 1.4. *Let J be an arbitrary nilpotent ring and let $1+J$ be the group associated to J. Then for all $m, n \in \mathbb{N}$:*

$$[1+J^m, 1+J^n] \subseteq [1+J, 1+J^{m+n-1}]. \tag{1}$$

This theorem plays an important role in the proof of Theorem 1.3.

2 Commutators in algebra groups

The main purpose of this section is to prove Theorem 1.4. First we show an easy lemma:

Lemma 2.1. *If $[1+A^k, 1+A^l] \subseteq [1+A, 1+A^{k+l-1}]$ for a nilpotent ring A, then $[1+B^k, 1+B^l] \subseteq [1+B, 1+B^{k+l-1}]$ for every quotient ring B of A.*

Proof. Let $\varphi : A \to B$ denote the natural ring homomorphism from A to B. Then we can extend this homomorphism to a group homomorphism $\overline{\varphi} : 1+A \to 1+B$ by the rule $\overline{\varphi}(1+a) := 1+\varphi(a)$. It is clear that $\overline{\varphi}(1+A^k) = 1+B^k$ for all k and $\overline{\varphi}([H,K]) = [\overline{\varphi}(H), \overline{\varphi}(K)]$ for all subgroups H, K of $1+A$. The assertion follows. □

Let $R = \mathbb{Q}$ or $\mathbb{Z}$ and denote by $F_R(X)$ the free algebra over R generated by the set X and by $F_R(n, X)$ the free nilpotent algebra over R with nilpotency class n generated by the set X. Then $F_R(n, X) \simeq F_R(X)/F_R(X)^n$. This means that $F_R(n, X)$ is the algebra of polynomials with noncommuting indeterminates in the set X subject to the relations that any product of n elements is zero. It is clear that every nilpotent ring is a quotient of $F_{\mathbb{Z}}(n, X)$ for some n and X, so by Lemma 2.1, in order to prove Theorem 1.4 it is enough to show formula (1) for the free nilpotent algebras over $\mathbb{Z}$. For the sake of simplicity we will denote a free nilpotent algebra (of unspecified class) over $\mathbb{Q}$ or over $\mathbb{Z}$ by $N(\mathbb{Q})$ or by $N(\mathbb{Z})$.
If J is a nilpotent algebra over the field R such that either $\operatorname{char} R = 0$ or $\operatorname{char} R = p$ and $x^p = 0$ for all $x \in J$ then we can define the map $\exp : J \to 1+J$ and the inverse of this map $\ln : 1+J \to J$ by the power series:

$$\exp(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^k}{k!} + \cdots,$$
$$\ln(1+x) = x - \frac{x^2}{2} + \cdots + (-1)^{k+1}\frac{x^k}{k} + \cdots.$$

The Campbell–Hausdorff formula says that for all $a, b \in J$:

$$\exp(a)\exp(b) = \exp(a + b + z(a,b)) \tag{2}$$

where $z(a,b)$ is an element in the Lie subalgebra generated by a and b. This formula can be found for example in [4, pp. 170–174] and it holds if either $\operatorname{char} R = 0$ or $\operatorname{char} R = p$ and $J^p = 0$.

Lemma 2.2. *Let J be a nilpotent algebra over $\mathbb{Q}$. Then the* exp *map establishes a bijection between J^k and $1+J^k$ for all k. Furthermore,* exp *is a bijection between the Lie commutator $[J^k, J^l]$ and the group commutator $[1+J^k, 1+J^l]$ for all k, l.*

Proof. The first part of the Lemma is obvious. Using formula (2) and the fact that $[J^k, J^l]$ is a Lie subalgebra of J it follows that $\exp[J^k, J^l]$ is a subgroup in $1 + J$. Let $x \in J^k$ and $y \in J^l$. Then $[\exp x, \exp y] = \exp([x, y] + w(x, y))$ by Lemma 9.15 in [5], where $w(x, y)$ is a rational combination of commutators in the elements x, y of weight ≥ 3. The inclusion $[1 + J^k, 1 + J^l] \subseteq \exp[J^k, J^l]$ is clear from this.
To see that $\exp(u) \in [1 + J^k, 1 + J^l]$ for any $u \in [J^k, J^l]$ we assume $k \geq l$ and we use reverse induction on k. Let $u = \sum_{i=1}^{n}[u_i, v_i]$, where $u_i \in J^k$ and $v_i \in J^l$. By the lemma mentioned before and by the Campbell–Hausdorff formula,

$$\exp(u)\Big(\prod_{i=1}^{n}[\exp(u_i), \exp(v_i)]\Big)^{-1}$$
$$= \exp(u)\prod_{i=n}^{1}\exp(-[u_i, v_i] - w(u_i, v_i)) = \exp(\omega),$$

where ω is a rational linear combination of commutators in the elements u_i, v_i of weight ≥ 3. Thus $\omega \in [J^{k+l}, J^l]$ and $\exp(\omega) \in [1+J^{k+l}, 1+J^l] \subseteq [1+J^k, 1+J^l]$ by reverse induction on k. Therefore $\exp(u) \in [1 + J^k, 1 + J^l]$ and we are done. □

Lemma 2.3. *If J is a free nilpotent algebra over $\mathbb{Q}$, then for all $k \geq 2$:*

$$[1 + J, 1 + J] \cap (1 + J^k) = [1 + J, 1 + J^{k-1}]. \tag{3}$$

Proof. If we apply the ln map to (3) then by Lemma 2.2 the equation

$$[J, J] \cap J^k = [J, J^{k-1}] \tag{4}$$

is equivalent to equation (3). Let n be the nilpotency class of J and let X be a free generator set of J. Then $B = \cup_{i=1}^{n-1} X^i$ is a basis of J, where $X^i = \{u_1u_2 \cdots u_i \mid u_j \in X,\ 1 \leq j \leq i\}$. Using that the Lie bracket is a bilinear function and X is a free generator set, it follows that $[J, J] \cap J^k$ can be generated (as a vector space) by the set

$$Y = \big\{[a, b] \mid a \in X^l, b \in X^m,\ l + m \geq k\big\}$$

Let $a = x_1x_2 \cdots x_l \in X^l$ and $b = y_1y_2 \cdots y_m \in X^m$ such that $l + m \geq k$. Then

$$\begin{aligned}[a, b] &= x_1 \cdots x_l y_1 \cdots y_m - y_1 \cdots y_m x_1 \cdots x_l\\ &= x_1 \cdot x_2 \cdots x_l y_1 \cdots y_m - x_2 \cdots x_l y_1 \cdots y_m \cdot x_1\\ &\quad + x_2 \cdot x_3 \cdots x_l y_1 \cdots y_m x_1 - x_3 \cdots x_l y_1 \cdots y_m x_1 \cdot x_2 + \cdots \in [J, J^{k-1}].\end{aligned}$$

Therefore $Y \subseteq [J, J^{k-1}]$ and $[J, J] \cap J^k \subseteq [J, J^{k-1}]$. It is clear that $[J, J] \cap J^k \supseteq [J, J^{k-1}]$, so the proof is complete. □

Remark 2.1. If J is any nilpotent algebra then equation (4) is not true in general. P. P. Pálfy showed me the following example: Let $J \leq M_4(\mathbb{Q})$ be the algebra of strictly upper triangular matrices with equal elements next to the main diagonal. Then

$$J = \begin{pmatrix} 0 & a & * & * \\ 0 & 0 & a & * \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad J^2 = \begin{pmatrix} 0 & 0 & b & * \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad J^3 = [J, J] = \begin{pmatrix} 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In this case $[J,J] \cap J^3 = J^3 \neq 0$ but $[J, J^2] = 0$.

Up till now we have worked with rational nilpotent algebras. Now we consider free nilpotent rings over $\mathbb{Z}$. Our next purpose is to prove (3) for $J = N(\mathbb{Z})$. First we prove an easy lemma.

Lemma 2.4. *Let V be a vector space over $\mathbb{Q}$ and let $B \subseteq V$ be a basis of V. For any subset Y we denote by $\langle Y \rangle_{\mathbb{Z}}$ the set of all linear combinations of elements from Y with integer coefficients. If $Y \subseteq \{a - b \mid a, b \in B\}$, then $\langle Y \rangle_{\mathbb{Q}} \cap \langle B \rangle_{\mathbb{Z}} = \langle Y \rangle_{\mathbb{Z}}$.*

Proof. Let $z = \sum_{i=1}^{m} \alpha_i y_i \in \langle Y \rangle_{\mathbb{Q}} \cap \langle B \rangle_{\mathbb{Z}}$ be such that $\{y_1, y_2, \ldots, y_m\} \subseteq Y$ is linearly independent, and each $\alpha_i \in \mathbb{Q}$ is nonzero. Then there is a minimal subset $B' \subseteq B$ such that $\{y_1, y_2, \ldots, y_m\} \subseteq \langle B' \rangle$. Since $m = \dim \langle y_1, y_2, \ldots, y_m \rangle < |B'|$, there is an element $b \in B'$ such that there exists exactly one y_k with nonzero coordinate in b. Then α_k is integer, thus

$$z - \alpha_k y_k = \sum_{i \neq k} \alpha_i y_i \in \langle Y \rangle_{\mathbb{Q}} \cap \langle B \rangle_{\mathbb{Z}} .$$

The result follows by induction on m. □

Lemma 2.5. *For all $k \geq 2$*

$$[1 + N(\mathbb{Z}), 1 + N(\mathbb{Z})] \cap (1 + N(\mathbb{Z})^k) = [1 + N(\mathbb{Z}), 1 + N(\mathbb{Z})^{k-1}].$$

Proof. It is evident that

$$[1 + N(\mathbb{Z}), 1 + N(\mathbb{Z})] \cap (1 + N(\mathbb{Z})^k) \supseteq [1 + N(\mathbb{Z}), 1 + N(\mathbb{Z})^{k-1}].$$

Since $1 + N(\mathbb{Z})^k \leq 1 + N(\mathbb{Q})^k$, it follows that

$$\begin{aligned} &[1 + N(\mathbb{Z}), 1 + N(\mathbb{Z})] \cap (1 + N(\mathbb{Z})^k) \\ &\qquad \leq [1 + N(\mathbb{Q}), 1 + N(\mathbb{Q})] \cap (1 + N(\mathbb{Q})^k) \\ &\qquad\qquad = [1 + N(\mathbb{Q}), 1 + N(\mathbb{Q})^{k-1}], \end{aligned}$$

using Lemma 2.3. If X is a free generator set of the free nilpotent algebra $N(\mathbb{Q})$ with nilpotency class n then we can write all elements of $N(\mathbb{Q})$ uniquely as polynomials in elements of X such that all terms of these polynomials have degree $< n$. Then $N(\mathbb{Z}) \leq N(\mathbb{Q})$ is exactly the set of polynomials with integer coefficients. So it is enough to prove that the elements of $[1+N(\mathbb{Q}), 1+N(\mathbb{Q})^{k-1}]$ with integer coefficients belong to $[1 + N(\mathbb{Z}), 1 + N(\mathbb{Z})^{k-1}]$. We define the degree of an element in $N(\mathbb{Z})$ as the smallest degree of its terms. Let $1 + z \in [1 + N(\mathbb{Q}), 1 + N(\mathbb{Q})^{k-1}] \cap (1 + N(\mathbb{Z}))$ be an arbitrary element. If z has degree $l > k - 1$ then $1 + z \in [1 + N(\mathbb{Q}), 1 + N(\mathbb{Q})^{l-1}]$ by equation (3). To prove that $1 + z \in [1 + N(\mathbb{Z}), 1 + N(\mathbb{Z})^{k-1}]$ we use reverse induction on l noting that the result is certainly true for l large enough.
Let $X^l = \{u_1 u_2 \cdots u_l \mid u_j \in X,\ 1 \leq j \leq l\}$ and $[X, X^{l-1}] = \{[u, v] \mid u \in X, v \in X^{l-1}\}$. We write $1+z = \prod [1+x_i, 1+y_i]^{\pm 1}$, where $x_i \in N(\mathbb{Q})$ and $y_i \in N(\mathbb{Q})^{l-1}$. It is clear that $[x_i, y_i] \in \langle [X, X^{l-1}] \rangle_{\mathbb{Q}} + N(\mathbb{Q})^{l+1}$, so

$$\begin{aligned} 1 + z = \prod [1 + x_i, 1 + y_i]^{\pm 1} &\in \Big(1 + \sum \pm [x_i, y_i] + N(\mathbb{Q})^{l+1}\Big) \cap (1 + N(\mathbb{Z})) \\ &\subseteq 1 + \Big(\langle [X, X^{l-1}] \rangle_{\mathbb{Q}} \cap N(\mathbb{Z})\Big) + N(\mathbb{Q})^{l+1}. \end{aligned}$$

But $\langle [X, X^{l-1}] \rangle_{\mathbb{Q}} \cap N(\mathbb{Z}) = \langle [X, X^{l-1}] \rangle_{\mathbb{Z}}$ by Lemma 2.4, henceforth $1+z \in 1+\langle [X, X^{l-1}] \rangle_{\mathbb{Z}} + N(\mathbb{Q})^{l+1}$. From this it follows that $z \in \sum_{j=1}^{m} \alpha_j [a_j, b_j] + N(\mathbb{Q})^{l+1}$, where $\alpha_j \in \mathbb{Z},\ a_j \in X$ and $b_j \in X^{l-1}$ for all j. Then

$$1 + z' := (1+z) \cdot (\prod [1 + \alpha_j a_j, 1 + b_j])^{-1}$$

is an element of $[1 + N(\mathbb{Q}), 1 + N(\mathbb{Q})^{k-1}] \cap (1 + N(\mathbb{Z}))$ and z' has degree greater than l. Thus $1 + z' \in [1 + N(\mathbb{Z}), 1 + N(\mathbb{Z})^{k-1}]$ by induction and so $1 + z \in [1 + N(\mathbb{Z}), 1 + N^(\mathbb{Z})^{k-1}]$, too. This completes the proof. □

Proof of Theorem 1.4

It was shown at the beginning of this section that it is enough to prove equation (1) in the case when $J = N(\mathbb{Z})$. It is clear that $[1 + N(\mathbb{Z})^m, 1 + N(\mathbb{Z})^n] \subseteq [1 + N(\mathbb{Z}), 1 + N(\mathbb{Z})] \cap (1 + N(\mathbb{Z})^{m+n})$. The right-hand side of this inclusion is exactly $[1 + N(\mathbb{Z}), 1 + N(\mathbb{Z})^{m+n-1}]$ by Lemma 2.5, so we are done.

3 Characters in algebra groups

Let $G = 1 + J$ be a finite algebra group over $\mathbb{F}_q$ where $q = p^f$ for some prime p. In this section we prove Theorem 1.2 and Theorem 1.3. We shall need the following result.

Lemma 3.1. *Let $G = 1 + J$ be a finite algebra group over $\mathbb{F}_q$ and $\chi \in \mathrm{Irr}(G)$. Then the following properties are equivalent:*

1. *There exist a proper algebra subgroup $H < G$ and $\varphi \in \mathrm{Irr}(H)$ such that $\chi = \varphi^G$.*
2. *χ_{1+J^2} is not irreducible.*

Proof. Suppose $H = 1 + U \neq G$ is an algebra subgroup and $\varphi \in \mathrm{Irr}(H)$ is such that $\chi = \varphi^G$. Let $K = H(1 + J^2) = 1 + U + J^2$. Then $1 + J^2 \leq K \neq G$ by Lemma 3.1 in [2]. Thus $\chi = (\varphi^K)^G$ and χ_K is not irreducible. Then χ_{1+J^2} is not irreducible, too.
Assume now that χ_{1+J^2} is not irreducible and let $\psi \in \mathrm{Irr}(1+J^2)$ be a constituent of χ_{1+J^2}. Let $H = 1 + U \geq 1 + J^2$ be a maximal $\mathbb{F}_q$-algebra subgroup such that ψ is extendible to H. Then $H \neq G$. We choose a $\varphi \in \mathrm{Irr}(H)$ such that φ is an extension of ψ and φ is a constituent of χ_H. Then for an arbitrary $x \in J \setminus U$ the subgroup $N_x = 1 + \mathbb{F}_q x + U$ is an $\mathbb{F}_q$-algebra subgroup and $|N_x : H| = q$. Let $\vartheta \in \mathrm{Irr}(N_x)$ be a character over φ, that is, φ is a constituent of ϑ_H. By Isaacs' Theorem 1.1, $\vartheta(1)$ and $\varphi(1)$ are both q-powers hence either $\vartheta_H = \varphi$ or $\vartheta = \varphi^{N_x}$. The first case cannot occur by the maximal choice of H, thus $\vartheta = \varphi^{N_x}$. Therefore $I_{N_x}(\varphi) = H$ for all x by Problem 6.1 in [1], thus $I_G(\varphi) = H$. Hence φ^G is irreducible by Problem 6.1 so $\chi = \varphi^G$. □

Proof of Theorem 1.3

Let $G = 1 + J$ be an algebra group and $\varphi \in \mathrm{Irr}(1 + J^2)$ be a G-invariant character. We prove by reverse induction that $[1 + J^2, 1 + J^k] \leq \ker \varphi$ for all $k \geq 2$. This is clear if $J^k = 0$. Assuming that $[1 + J^2, 1 + J^k] \leq \ker \varphi$ it follows that $1 + J^k \leq Z(\varphi)$, where $Z(\varphi)$ denotes the center of φ. Hence

$\varphi_{1+J^k} = \lambda \cdot \varphi(1)$, where λ is a G-invariant linear character of $1 + J^k$. It follows that $[1 + J, 1 + J^k] \leq \ker \varphi$. Using Theorem 1.4 we have $[1 + J^2, 1 + J^{k-1}] \leq [1 + J, 1 + J^k]$, thus $[1 + J^2, 1 + J^{k-1}] \leq \ker \varphi$.

Proof of Theorem 1.2

If $\chi \in \mathrm{Irr}(G)$ is not linear then χ_{1+J^2} is not irreducible by Theorem 1.3. By Lemma 3.1 there exist an algebra subgroup $H \neq G$ and $\varphi \in \mathrm{Irr}(H)$ such that $\chi = \varphi^G$. Using induction on $|G|$, we obtain that there exist an algebra subgroup $L \leq H$ and $\lambda \in \mathrm{Irr}(L)$ such that λ is a linear character of L and $\varphi = \lambda^H$. The theorem follows from the transitivity of induction.

Acknowledgements. The author is very grateful to P. P. Pálfy and to the anonymous referee for their helpful comments.

References

[1] I. M. Isaacs, *Character theory of finite groups*, Dover, New York, 1994.

[2] I. M. Isaacs, Characters of groups associated with finite algebras, J. Algebra **177** (1995), 708–730.

[3] Carlos A. M. André, Irreducible characters of finite algebra groups, *Matrices and group representations (Coimbra, 1998)*, 65–80, Textos Mat. Sér. B, 19, *Univ. Coimbra, Coimbra,* 1999.

[4] N. Jacobson, *Lie Algebras*, Interscience Publishers, New York, 1962.

[5] E. I. Khukhro, *p-automorphisms of finite p-groups*, Cambridge University Press, Cambridge 1997.

Zoltán Halasi
Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
P. O. Box 127
H-1364 Budapest
Hungary
e-mail: haca@cs.elte.hu